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THE PARETO-LÉVY LAW AND THE DISTRIBUTION OF INCOME*

BY BENOIT MANDELBROT[1]

SUMMARY

THE PURPOSE of this paper is twofold. On the one hand, we wish to give an account of a set of new models for certain distribution properties of an important class of economic quantities, which includes "income" (see [15], [17]). On the other hand, we think that the tools which we shall use are as important as the results which we hope to achieve: that is, we intend to draw the economist's attention to the great potentital importance of *"stable non-Gaussian"* probability distributions. To give a sharper focus to our paper, we shall develop our main points within the frame of a theory of income distribution; but our approach will be immediately translatable in terms of similar quantities, and our theory may well be more reasonable, or the empirical fit better, in the case of some other quantities. We may thus paraphrase what a famous author said of Brownian motion: it is possible that the properties which we shall study are identical to those of income; however, the information available to us regarding incomes is so lacking in precision that we cannot really form a judgment on the matter.

This paper will be almost exclusively theoretical. Our point of departure will be an interesting empirical observation, namely, that over a certain range of values of income U, its distribution is not markedly influenced *either* by the socio-economic structure of the community under study, *or* by the definition chosen for "income." That is, these two elements may at most influence the values taken by certain parameters of an apparently universal distribution law. This law was discovered by V. Pareto in 1897 [24]; actually, several different statements have at times been called "Pareto's law." In the introduction we shall very carefully distinguish between these statements, and we shall comment upon some existing theories of income distribution.

* Manuscript received June 19, 1959.

[1] The work in this paper was started while the author was associated with the University of Geneva (under partial support of the Rockefeller Foundation), then with the University of Lille and the École Polytechnique in Paris.

We shall then introduce a new version of Pareto's law which belongs to the class of "stable" distributions, and which we shall designate as *"the Pareto-Lévy law"* (P–L).[3] This law also explains partially the data relative to the middle-income range.

The initial motivation for this law will be essentially twofold. *First*, among all the possible interpolations of the weak (asymptotic) Pareto law, the P–L law is the only one which strictly satisfies one (strong) form of the condition of invariance of the distribution, relative to the definition of income. *Second*, a P–L distribution is a possible limit distribution of sums of random variables. That is, as we shall demonstrate, *the Gaussian distribution is not the only possible such limit* (as is too generally assumed), and it is unnecessary (as well as insufficient) to try to save the limit argument by applying it to $\log U$, instead of U, as is done in some theories leading to the "log-normal law" for U.

The P–L law turns out to have many other very desirable properties. *It has also the drawback that it is limited to the interpolation of Pareto's law, when its index* α *is such that* $1 < \alpha < 2$, (see 1.1 for a definition of α).

In a series of other articles, which has started with [18], [19], [20], we also study the P–L *stochastic processes*, a family of models of income variation. These processes again preserve certain desirable properties of the classical Gaussian processes without such drastic defects as the fact that income variation cannot possibly be Gaussian. It turns out that *in the high income range, a* P–L *Markovian process can be approximated by a random walk of* $\log U$: that is, we are able to derive the fundamental initial assumptions of a model due to Champernowne [2]. Moreover, a P–L Markovian process behaves quite reasonably in the intermediate range of incomes.

1. INTRODUCTION

1.1. *The strong Pareto law.* Let us begin with a basic variant, which will be referred to as the *strong Pareto law*. Let $P(u)$ be the percentage of individuals with an income U (over some fixed period of reference) exceeding some number u (u is assumed to be a continuous variable). The strong Pareto law asserts that:

$$P(u) = \begin{cases} (u/u^0)^{-\alpha} & \text{when } u > u^0 \\ 1 & \text{when } u < u^0 . \end{cases}$$

[3] This associates Pareto's name with that of Paul Lévy, who first studied the stable distributions intensively. (See in particular Lévy [10], [11], [12].) To the best of our knowledge the application of these laws to economics is entirely new as well as are the definition and properties of the "Pareto-Lévy" processes of [18], [19], [20].

Then, the "density" $p(u) = -dP(u)/du$ satisfies

$$p(u) = \begin{cases} \alpha(u^0)^\alpha u^{-(\alpha+1)} & \text{when } u > u^0 \\ 0 & \text{when } u < u^0 . \end{cases}$$

This distribution is fully specified by *two "state variables"*: u^0, a scale factor, and α, which will turn out to be some kind of index of inequality of distribution. The strong law leaves the value of α undetermined (although in some cases, one immediately states that $\alpha > 1$). We may also find the stronger statement that $\alpha = 3/2$: the corresponding variant of the strong law will be referred to as the *strongest Pareto law*. Graphically, the strong law implies that the ("double logarithmic") graph of $y = \log P$, as a function of $v = \log u$, is a straight line.

The empirically observed $P(u)$ are of course percentages relative to finite populations. They will further be considered as frequencies relative to a random sample, drawn from an infinite population. [Empirically, the law holds as well for samples of a few hundred (the "burghers" of a city-state of the Renaissance) or of close to a hundred million (the taxpayers in the USA).] That is, U will be treated as a random variable with values u, and the curve describing the variation of U in time will be treated as a random function.

1.2. *The weak Pareto law.* The strongest Pareto law is now acknowledged to be empirically unjustified (and it should be noted that many purported "disproofs" of "the" Pareto law apply to this variant only). On the contrary, there is little question of the validity of the Pareto law if "sufficiently" large values of u are concerned. The best way of taking care of the limitations of the strong law is simply to state that:

$$P(u) \text{ "behaves like" } (u/u^0)^{-\alpha}, \text{ as } u \to \infty .$$

This is the *weak Pareto law*. Such a statement is useful only if the exact definition of "behaves like" conforms to the empirical evidence and (taking advantage of the indeterminacy of such evidence) lends itself to easy mathematical manipulation. The following definition of "behaves like" is usually found to be adequate:

$$P(u) \sim (u/u^0)^{-\alpha} .$$

That is:

$$\frac{P(u)}{(u/u^0)^{-\alpha}} \to 1, \text{ as } u \to \infty ,$$

or,

$$P(u) = \{1 + \epsilon(u)\}(u/u^0)^{-\alpha}, \text{ where } \epsilon(u) \to 0, \text{ as } u \to \infty .$$

Graphically, this means that the curve $(\log P, \log u)$ is asymptotic to the straight line which represents the strong Pareto law.

1.3. *Interpolation of the weak Pareto law.* The range of values of u which is accounted for by Pareto's analytic expression covers only a part of the total population. Elsewhere the "density" $-dP(u)/du$ is represented by a quite irregular curve, the shape of which depends in particular upon the breadth of coverage of the data considered. This has been particularly clearly emphasized by H. P. Miller [22]. He has shown that the income distribution is skewed and presents several maxima, if one includes all individuals, even those with no income and part-time workers, and if one combines the incomes of men and women. However, the separate income distributions of most of the different occupational categories, as distinguished by the census, are both regular and fairly symmetric; the main source of skewness in the overall distribution and hence in Pareto's law can be traced to the inclusion of self-employed persons and managers together with all wage earners. One may also note that the method of reporting income differs by occupational categories, and that, as a result, the corresponding data are not equally reliable.

The above reasons make it *unlikely that a single theory could ever explain all the features of the income distribution* or that a single empirical formula could ever represent all the data. As a result, it has been frequently suggested that several different models may be required to explain the empirical $P(u)$; unfortunately, we know of no empirical check of this conjecture. In any case, the present paper will be devoted almost exclusively to a theory of high income data and of the weak Pareto law. It is unlikely that the interpolation of the results of our model will be able to explain finally all the middle income data. However, we shall not examine this point in great detail.

1.4. *Two laws of income distribution which are in contradiction with the weak Pareto law.*

A. Pareto himself has suggested the following possible improvement in his law:

$$p(u) = -dP/du = ku^{-(\alpha+1)}\exp(-bu) \quad (\text{where } b > 0).$$

However, the weak Pareto law must be at least approximately correct for large u. Hence, the parameter b must be very small, and the evidence against the hypothesis that $b = 0$ is not strong. The choice, therefore, between the hypotheses $b = 0$ and $b \neq 0$ may be legitimately influenced by ease of mathematical manipulation and ex-

planation. We shall see that the weak Pareto law ($b = 0$) has some crucial properties, which will be used as a basis for a theory, and which disappear if $b \neq 0$. We shall therefore disregard the case $b \neq 0$ (the more so since the formula above provides no improvement for middle values of u). (See 2.4 for a further discussion of $b = 0$.)

B. The best known statement about income distribution, apart from the weak Pareto law, is the *log-normal law* [8], [1], which claims that the variable $\log U$ [or perhaps better $\log(U-u')$, where $u'>0$] is well represented by the Gaussian distribution. The empirical evidence for this is that the graph of (P, u) on log-normal paper seems to be straight. Such a graph emphasizes a different range of values of u from that of the Pareto graph, so that the graphical evidence for the two laws is not contradictory. The motivation for the log-normal law is, however, largely theoretical (see 1.5). Roughly speaking, a log-normal U can be explained by assuming that $\log U$ is the sum of many additive components.

1.5. *Some existing models of income distribution considered as "thermodynamic" theories.* There is a great temptation to consider the exchanges of money which occur in economic interaction as analogous to the exchanges of energy which occur in physical shocks between gas molecules. In the loosest possible terms, both kinds of interactions "should" lead to "similar" states of equilibrium. That is, one "should" be able to explain the law of income distribution by a model similar to that used in statistical thermodynamics: many authors have done so explicitly, and all the others of whom we know have done so implicitly.

Unfortunately, the Pareto $P(u)$ decreases much more slowly than any of the usual laws of physics, so that if one wants to apply the physical theory mechanically, one must somehow argue that U is a less *intrinsic* variable than some slowly increasing function $V(U)$. For that, one must renounce the additive properties of U. The seemingly universal choice for V is $V = \log U$ [or $V' = \log(U - u')$]. This choice is suggested by the fact that one plots $\log u$ empirically. But it can also be traced back to the "moral wealth" of Bernouilli, and it apparently can be justified by some law of proportionate effect, or by some kind of Weber-Fechner law. If this choice of V is granted, one has to explain the normal law for the middle zone of v's, and the exponential law $P(v) = \exp\{-\alpha(v - v^0)\}$ for the upper zone of v's.

Indeed, many existing models of the Pareto distribution are reducible to the observation that (for any α) $\exp(-\alpha v)$ *is a possible barometric density distribution in the atmosphere.* Alternatively, con-

sider a set of Brownian particles, floating in a gas at a uniform temperature and density, the whole being enclosed in a semi-infinite tube with a closed bottom and an open top; assume further that the field of gravity is uniform. Then the density distribution which the Brownian particles attain as a state of final equilibrium is the exponential. This is the result of a compromise between two forces, both uniform along the tube, i.e., the force of gravity, which alone would tend to pull all particles to the bottom, and the influence of heat motion, which alone would tend to diffuse them all to infinity. Clearly, the models of income distribution we are now considering involve an interpretation of the forces of diffusion and of gravity.

Unfortunately, such interpretations are never sufficiently intuitive to exclude, for example, the removal of the counterpart of the bottom of the tube. The trouble is that such a removal changes everything. In fact, there is no steady limit state any longer, because all the Brownian particles diffuse to minus infinity. It is true that the Gaussian distribution does appear as a *conditional* distribution for V (this distribution is valid if one assumes that all the particles start from the same point), but this cannot really be a basis for a model of the log-normal law.

This difficulty is already very clear in the simplest model of a diffusion, the so-called *random walk*, in which both time and V are quantified. Time is integral and V is a multiple of a unit v'. This model (apparently introduced into economics by Champernowne [2]) further assumes (1) that the variation of V is Markovian, that is, $V(t+1)$ is a function only of $V(t)$ and of chance, and (2) that the probability that $V(t+1) - v(t) = kv'$ becomes independent of $v(t)$ as v increases. Additional assumptions are required to get either Pareto's law or the log-normal law, and the difference between these additional assumptions has no intuitive meaning, which makes both conclusions unconvincing. However, the models which lead to the exponential are still slightly better: in fact, we can, perhaps, argue that the apparent normality of the "density" $p(v)$ in the central zone of v's simply means that $-\log p(v)$ may be represented by a parabola in that zone, whereas for large v's, it is represented by a straight line. Such a parabolic interpolation needs no limit theorems of probability for its justification; it applies to any regularly concave curve, at least in the first approximation.

In the case of models of the weak Pareto law, further non-intuitive assumptions are necessary to rationalize the empirical fact that $\alpha > 1$ in all cases.

We shall not analyze any of the other existing translations into economic terms of various models of the normal or exponential distributions, which occur in statistical thermodynamics, such as gas equilibrium or Brownian motion. Because of their common "thermodynamic" character, these theories are essentially equivalent. In particular, they do not assume any institutional income structure; all income recipients are treated as entrepreneurs. A rewording of the classical theories, however, can make them applicable to many possible institutional structures of wage and salary earners; see Lydall [13][8].

In this article (see also [18]) we shall attempt to show that *the analogy of statistical physics need not be abandoned, if we want to avoid the weaknesses which mar existing theories.* We shall not require the transformation $V = \log U$, and we shall not try to justify the introduction of the economic counterparts of such conditions as the presence or the absence of a bottom to the tube in which Brownian motion is studied. That is, we shall not try to force income into the structure of statistical thermodynamics (even implicitly); rather, we shall attempt explicitly to generalize the statistical methods of thermodynamics, in order to cover the economic concept of income.

We shall be successful only in the case where $1 < \alpha < 2$, a restriction which we shall discuss repeatedly.

2. PARETO-LÉVY RANDOM VARIABLES

2.1. *An analysis of the definition of income.* One of the principal claims of this paper is that it is impossible to "explain" why Pareto's law, and not some other law, is satisfied by income distributions, without also studying the inseparable problem raised by the fact that *essentially the same law continues to be followed by the distribution of "income," despite changes in the definition of this term.* Such an invariance is of course very important to census analyzers, because of the small effect of large changes in methods of estimating income. To study this problem one needs a practical way of expressing the different definitions of U. We shall argue that there are several ways of distinguishing different sources of U. Therefore, U may be written in different ways as a sum of elements, such as: A, agricult-

[8] We had indenpendently rediscovered this model and had discussed it in the original text of this paper, as submitted on June 19, 1959. The coincidence of the predictions of the P-L model and of the Lydall model for $1<\alpha<2$, shows that the distribution which corresponds to the least amount of organization could be "frozen" without modifying it, reinterpreted as a wage distribution and then let to evolve along conceptually quite different lines. This fact has great relevance to the problem of the value of α and to the fact that α has recently tended to increase beyond 2, in highly organized industrial societies.

ural, commercial, or industrial incomes; B, incomes in cash or in kind; C, ordinary taxable income or capital gains; D, incomes of different members of a single taxpaying unit, etc... Let the income categories in a certain decomposition of U be labeled as U_i $(1 \leq i \leq N)$. We shall assume that every method of reporting or estimating income corresponds to the observation of the *sum* of the U_i corresponding to some subset of indices i. This quite reasonable consideration imposes the restriction that the scale of incomes themselves is more intrinsic than any transformed variables, such as $\log U$.[4]

Let the incomes in the categories U_i be considered as statistically independent. This assumption idealizes the actual situation. *A priori*, this abstraction may seem a bad first approximation, since when income is divided into two categories U' and U'', such as agricultural and industrial incomes, the observed values u' and u'' are usually very different. We shall find, however, that in the weak Pareto case, when the parts are independent, one would actually expect them to be very unequal; hence, although such an inequality cannot be a confirmation of independence, at least it does not contradict it in this case.

Under these assumptions, the only probability law for income which could possibly be observed must be such that if U' and U'' follow this law (up to a scale transformation, and up to the choice of origin), then $U' \oplus U''$ must also follow the same law (the sign $\oplus$ designates the addition of random variables). That is, whatever $a'>0$, b', $a''>0$, b'', there exist two constants $a > 0$ and b, such that

$$(a'U + b') \oplus (a''U + b'') = aU + b\,.$$

Such a probability law is said to be "stable" (under addition), and its density as well as the variable U itself are "stable." *The family of all laws which satisfy this requirement* has been constructed by Paul Lévy [10]: it includes, naturally, the Gaussian law, but it also *includes certain non-Gaussian laws*, each of which turns out to satisfy the weak Pareto law, with some $0 < \alpha < 2$. In other words, *the additivity property of U, and the behavior of $P(u)$*, under the weak Pareto law (both of which disappear if the scale of U is changed) *turn out to be precisely sufficient and necessary for the application of Lévy's theory of stable laws.*

Further, the parameters of a non-Gaussian stable law can be arranged so that $E(U) < \infty$, which requires $1 < \alpha < 2$, and so that $p(-u)$ decreases very much faster than $p(u)$, when $u \to \infty$: in the

[4] The same kind of division may also be performed in the direction of time. The problem arises that it is unlikely that the year be an intrinsic unit of time. If it is not, how can a single law hold irrespective of the time unit chosen? See [18], [19], [20].

latter case, the stable law may be called *positive*, an abbreviation for "maximally skewed in the *positive* direction."

Since the usual terminology of probability (stable distributions) does not serve our purpose, *we propose to refer to "positive" stable distributions with* $1 < \alpha < 2$, *as "Pareto-Lévy distributions."*

It is not strictly true that the distribution of income is invariant over the whole range of u, with respect to a change in definition of income. We argue, however, that *if the variable U is not Gaussian and is very skewed, the most reasonable "first order" assumption about income is that it is a* P–L *variable.*

The density $p(u)$ of the P–L law unfortunately cannot be expressed in a closed analytic form, but is determined by its bilateral Laplace transform (valid for $b > 0$):

$$G(b) = \int_{-\infty}^{\infty} \exp(-bu)\,|\,dP(u)\,| = \int_{-\infty}^{\infty} \exp(-bu)p(u)du$$
$$= \exp\{(bu^*)^\alpha + Mb\}\ ,$$

which depends upon three parameters: α $(1 < \alpha < 2)$, u^* (which is a positive scale parameter), and M (which is identical to $E(U)$).

$G(b)$ yields the result:

$$P(u) \sim u^{-\alpha}[u^*\Gamma(1-\alpha)]^\alpha \qquad \text{for } u \to \infty\ ,$$

where $\Gamma(1-\alpha)$ is the Euler gamma-function.

For other values of u, the Pareto-Lévy distribution had to be computed. A few sample graphs of densities appear in Figure 1. More detailed tables will be published by Mandelbrot and Zarnfalier [21].

We see that as long as α is not close to 2, the P–L density curve very rapidly becomes indistinguishable from a strong Pareto curve of the same α. [For this, the origin of the strong Pareto curve must be chosen properly and one must have $u^\circ = u^*\Gamma(1-\alpha)$.] The two curves converge near $u = E(U)$, when α is in the neighborhood of 3/2, and at even smaller values of u, when α is less than 3/2: that is, the asymptotic behavior of $P(u)$, derived from $G(b)$, is very rapidly implemented. As for large negative values of u, by the P–L law they have a probability which is not zero, but decreases so rapidly as $u \to -\infty$, that they may be safely disregarded; one finds that

$$\log[-\log p(u)] \sim \frac{\alpha}{\alpha - 1} \log|u|\ .$$

This is a faster decrease than in the case of the Gaussian density: see Appendix I.

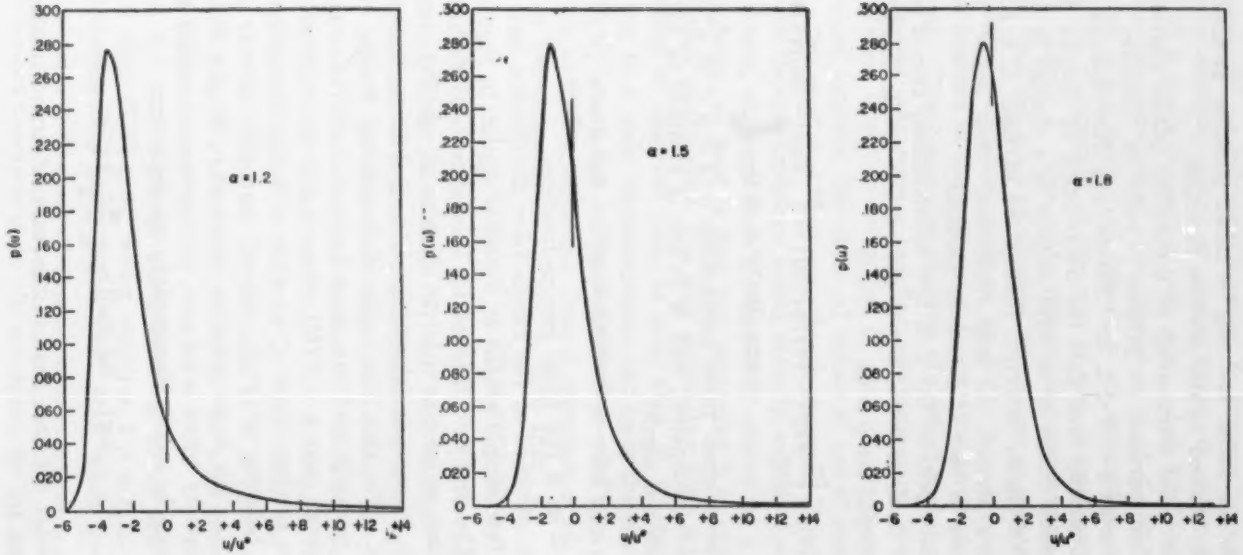

FIGURE 1: DENSITIES OF REDUCED P-L VARIABLES, ROF $M=0$ AND $\alpha=1.2$, 1.5, 1.8

Finally, near $E(U)$ and the most likely value of U, the P–L curve has the kind of skewness which one finds in the empirical data and which one hopes to derive in a theoretical curve.

Let α now approach 2. At the limit, the P–L density will tend toward a Gaussian density. Close to the limit, the P–L density already resembles a Gaussian one. Only for large values of u, (which have a very small probability) is the Gaussian decrease of $P(u)$ replaced by a Paretian decrease.

In other terms, the P–L density is worth considering only if α is not "too close" to 2. Even in this range, however, the prediction for intermediate values of u is quite inadequate to cover the complete curve of incomes, as described for example by Miller [22]. But as part-time and unskilled workers are eliminated we more nearly approximate such a curve. On the other hand, the income distribution of unskilled workers is fairly symmetric and has a fairly small dispersion.[8] Since the distribution of unskilled wages differs from that of high incomes, the P–L law would *a priori* explain only the latter incomes. If we examine any income category as defined by the Bureau of Census, we can seldom tell in advance whether its income mechanism is closer to that of the unskilled workers or to that of the largest income recipients. Therefore, if we wish to determine how wide are the categories to which the P–L law applies, the only solution is to start from the P–L interpolation of high incomes, and then to see how many other incomes (and which) must be added to obtain a good fit. This establishes a distinction between two kinds of income categories. The reasonableness of this distinction should be checked on independent grounds. For the time being we shall be content to observe that for α far enough from 2 the prediction made by the P–L law is "reasonable." (cf. 2.6.)

We shall conclude this section with a few statements concerning non-P–L stable distributions. *The Gaussian distribution* is of course stable, and it *is the only stable distribution with a finite variance.* Further, all the stable variables with a finite mean are differences of P–L variables scaled by arbitrary positive coefficients. The bilateral generating function is no longer defined, and we must consider the usual characteristic function (c.f.). For a P–L variable, the c.f. is immediately obtained as

$$\varphi(\zeta) = \int_{-\infty}^{\infty} e^{iu\zeta}\, p(u)du = \exp\left[- iM\zeta - |\zeta|^{\alpha}(u^*)^{\alpha}\left\{1 - \frac{i\zeta}{|\zeta|}\operatorname{tg}\frac{\alpha\pi}{2}\right\}\cos\frac{\alpha\pi}{2}\right].$$

[8] We may even think that their incomes would become non-random if their occupations were entirely interchangeable and if their changes in type of employment were without obstruction and costless.

Hence, denoting by $(1+\beta)/(1-\beta)$ the ratio of the positive and negative components, the general stable variable, with $E(U) < \infty$, has a c.f. of the form:

$$\varphi(\zeta) = \exp\left[- iM\zeta - |\zeta|^{\alpha}(u^*)^{\alpha}\left\{1 - \frac{i\beta\zeta}{|\zeta|}\operatorname{tg}\frac{\alpha\pi}{2}\right\}\cos\frac{\alpha\pi}{2}\right]$$

where $u^* \geq 0$, $|\beta| \leq 1$ and $1 < \alpha < 2$.

(We may note that the same formula, with $0 < \alpha < 1$, gives another family of stable variables, with $E(U) = \infty$. The stable variables which do not fall into either of the above families are those of Gauss and of Cauchy and finally a small family of variables connected with those of Cauchy.)

2.2. *Another (equivalent) property of the stable laws.* The stable laws have another (equivalent) property: they are the only possible limit laws of weighted sums of identical and independent random variables.[6] That is, if U is decomposed into a sum of a *large* number of components U_i, we need not resort to the above argument of observational invariance; we know that (1) if the sum is not Gaussian (which is a conspicuous fact), (2) if the expected value of the sum is finite (which is also a fact), and (3) if the probability of $-u$ is much less than that of u, for $u > 0$ and large, then *no hypothesis about the distribution of the parts is necessary to conclude that the sum can only be a Pareto-Lévy variable.* We shall develop these points in more detail, and in 2.6 we shall show that the above restrictions may be reduced to broad "qualitative" properties, such as the likelihood that two independent variables U' and U'' contribute unequally to the sum $U' \oplus U''$.

[6] Every stable law is such a limit. To prove this, assume that the variables U_i themselves are stable; by induction of stability, the $\oplus$-sum $\sum_{i=1}^{N} U_i$ will be a variable of the form $a(N)U + b(N)$ and hence $\{a(N)\}^{-1}\{(\sum_{i=1}^{N} U_i) - b(N)\}$ will have the same distribution function as U. Conversely, assume that a certain normed sum of the variables U_i has a limit. Write the N-th normed sum, taken in the sense of $\oplus$, as:

$$A(N)\sum_{i=1}^{N} U_i - B(N)$$
$$= \frac{A(N)}{A(n)}\left\{A(n)\sum_{i=1}^{n} U_i - B(n)\right\} \oplus \frac{A(N)}{A(m)}\left\{A(m)\sum_{i=1}^{m} U_i - B(m)\right\} + C(N)$$

where $N = n + m$. By letting both n and m tend to infinity, we find that the definition of a stable variable must be satisfied by the limit of $A(N)\sum_{i=1}^{N} U_i - B(N) + C(N)$.

We should also mention the following *theorem of Gnedenko.* For the convergence of $N^{-1/\alpha}\sum_{i=1}^{N} U_i - B(N)$ to a stable limit, it is *necessary and sufficient* that $0 < \alpha < 2$, and that for $u > 0$, $F(u) = 1 - P(u) = 1 - \{1 + \varepsilon'(u)\}(u/u_1)^{-\alpha}$, where $\varepsilon'(u) \to 0$, as $u \to \infty$, and for $u < 0$, $F(u) = \{1 + \varepsilon''(u)\}(u/u_2)^{-\alpha}$, where $\varepsilon''(u) \to 0$, as $u \to -\infty$.

It should be noted that an explanation of the distribution of income in terms of the addition of many contributions does not involve the $\log U$ transformation. This avoids the usual limit argument, which leads to the log-normal law and contradicts Pareto's law.

These results may be better than expected because the limit distribution might not hold for U, which is a sum of *only a few* random components. That is, if one wants to increase the number of components of U, one must abandon the hypothesis of independence at some point: the more U is subdivided the less independent the components become. This is a difficulty which is not proper to the problem of income, but is acutely present in all social science applications of probability theory.[7] Of course, the principle of the difficulty appears in physics for example in explaining why some kind of noise is Gaussian. But in most physical problems there exists a sufficiently large zone between the systems which are so small that they are impossible to subdivide and those which are so large that they can no longer be considered as homogeneous. No such zone exists in most problems of economics, so that a seemingly successful application of a limit theorem may seem too good to be true.

To sum up, it cannot be strictly true that the additive components are independent and have the same distribution (up to scale). However, the "likelihood" of any distribution is greatly increased if this is the only one reducible to limit arguments. Hence, *if a sum of many components is not Gaussian, is skewed, and such that* $E(U) < \infty$, *then the most reasonable first assumption concerning the sum is that it follows the Pareto-Lévy law.*

More precisely, two extreme interpretations are possible. The "minimal" interpretation regards the *Pareto-Lévy distribution as just another law which is correct asymptotically, is sufficiently easy to handle, and is useful in the first approximation* (after all, the Gaussian itself is frequently a good first approximation to distributions which, actually, are certainly not Gaussian).

At the other extreme we may *take the Pareto-Lévy law entirely seriously*, and try to check its ability to predict some properties of income distribution which otherwise would seem independent of the weak Pareto law. We think that such predictions were in fact achieved. This provides some claim for a "maximal" interpretation of the P–L law, which regards its invariance and limit properties as being "explicative."

2.3. *Stable laws are very well known in abstract probability theory*

[7] In estimating a mean by an empirical average, often the sample is too small or the many available data are strongly correlated.

(cf. Lévy [10], [11], [12], Gnedenko and Kolmogoroff [8]). We might, therefore, have introduced the P–L law without special motivation other than the fact that the known asymptotic behavior of $P(u)$ and its recently computed behavior for intermediate u make it an attractive interpolation for the income data (Mandelbrot [18]). Unfortunately, the behavior of the P–L law is in many other ways quite different from that to which most statisticians are accustomed from their constant handling of the Gaussian law. The only way of really showing how far this law is suited to the present problem is to sketch its theory, starting with a heuristic introduction to its properties (from the viewpoint of addition and of the study of extremal values), and ending with some rigorous results.

We see how the three main drawbacks of the classical theories vanish. Many variants of the same argument all lead to the same result; no change of scale of U is necessary, the behavior of $P(u)$ being *exactly* what is needed, and α is "near" 3/2. In fact, α should not be too close to 2, and cannot be greater than 2, a point which requires some elaboration.

2.4. *Concerning the sign of* $\alpha-2$ *and the variance of* U. A few empirical data claim to lead to $\alpha > 2$. To explain this by a limit argument requires most specific and unlikely assumptions. Similarly, the observational invariance of the stable distribution cannot hold if $\alpha > 2$, unless it is weakened (2.5) to require that the sum of *a few* weak Pareto variables with $\alpha > 2$ is a weak Pareto variable with $\alpha > 2$; as the number of addends increase (2.8), the weak Pareto law holds for a decreasing zone of values of u and finally the sum tends towards a Gaussian variable.

We find that the sign of $\alpha - 2$ distinguishes two sets of different random variables. The occurrence of $\alpha > 2$, or even $\alpha < 2$ but close to 2, may mean a predominance of salaries (Lydall's income model [13] is valid for all values of α). (Another explanation may be found in Mandelbrot [19].) On the other hand, several of the examples where $\alpha > 2$ are encountered occur in communities of Oceania which are very much less self-contained than, for example, Great Britain or the U.S.A. Their distributions of U may perhaps be truncated. In any case, the sign of $\alpha - 2$ raises an important empirical problem: Is the fit of the weak law equally good regardless of this sign, and does this law represent a higher percentage of the data where $\alpha < 2$?

A second consequence of the limitation $1 < \alpha < 2$ is that *the second moment of* U *is then infinite*, the first moment being finite. This is easily seen, because by integration by parts,

$$E(U) = \int_{-\infty}^{\infty} u dF(u) = -\int_{-\infty}^{\infty} u dP(u) = \int_{-\infty}^{\infty} P(u)du < \infty$$

and $D(U) = E(U^2) - [E(U)]^2$ where

$$E(U^2) = \int_{-\infty}^{\infty} u^2 dF(u) = \int_{-\infty}^{\infty} u^2 dP(u) = 2\int_{-\infty}^{\infty} uP(u)du = \infty .$$

(We assume that the behavior of $P(u)$ for negative u does not lead to a convergence problem for $E(U)$.)

The last result holds for both strong and weak Pareto variables if $1 < \alpha < 2$, but it would, for example, cease to hold for the density $p(u) = ku^{-(\alpha+1)} \exp(-bu)$ of 1.4. This provides a new and important test of our conjecture that $b = 0$. The finite $E(U)$ means that if u_i are samples from a Pareto distribution, the empirical mean $E_N = \sum_{i=1}^{N} u_i/N$ tends to $E(U)$ with probability 1 (Kolmogoroff's strong law of large numbers); in addition, E_N is a good estimate for $E(U)$, if N is large. Now consider $D_N = \sum_{i=1}^{N} (u_i - E_N)^2/N$. If $\alpha > 2$, D_N tends to a finite limit $D(U)$, of which it is a good estimate, and $\sum_{i=1}^{N} (u_i - E_N)/\sqrt{N}$ tends to a Gaussian variable. This result changes little if one adds an exponential factor with small b. However, if $\alpha < 2$, and $b = 0$ the *limit* of D_N is *infinity* (one may show that D_N grows without limit like $N^{2/\alpha-1}$) but if b becomes > 0, $D(U)$ is finite. Therefore, the usefulness of the exponential factor $\exp(-bu)$ may be tested by checking whether D_N keeps increasing with N in the case of the largest of the samples available. We could not make the direct test, but an indirect test results from the following observation: the ordering of the different populations by "increasing inequality" should presumably be identical with their ordering by decreasing α; on the other hand, more usual measures of inequality are given by D_N, D_N/E_N or $\sqrt{D_N}/E_N$. These two methods of ordering populations have been compared and found entirely contradictory. This result ceases to be absurd if one takes account of the fact that the values of N in the different samples which were compared range from 10^3 to 10^6; and it seems that even if $D(U)$ were finite, it would not be approached, even with the largest samples. In that case, *irrespective of any theory*, it is preferable to take $b = 0$, and $D(U) = \infty$. Further, no function of D_N is adequate to compare degrees of inequality, except perhaps between samples of identical size.

Another test of the usefulness of the approximation $D = \infty$ is provided by the relative contribution to D_N of the largest of the u_i: this is very large (close to 1/2 for the Wisconsin incomes [23]), as might be predicted from the theory of the Pareto-Lévy law. The usual procedure

of truncating U in order to avoid $D = \infty$ distorts the whole problem.

2.5. *Heuristic study of the addition of two independent random variables, particularly in the Gaussian and weak Pareto cases.* Let U' and U'' be two independent random variables, having the same probability density $p(u)$. We shall compare the behavior of $p(u)$ for large u, and that of $p_2(u) = \int_{-\infty}^{\infty} p(x)p(u-x)dx$, which is the density of the sum $U = U' \oplus U''$. It will be assumed that u may vary to $+\infty$.

One basic case is

$$p(u') = \begin{cases} C \exp(-bu') & \text{if } u' \geq 0, \\ 0 & \text{if } u' < 0. \end{cases}$$

Then,

$$p_2(u) = \int_0^u C^2 \exp(-bx) \exp[-b(u-x)]dx$$
$$= \int_0^u C^2 \exp(-bu)dx = C^2 u \exp(-bu).$$

Note that in this case $\log[p(u')] = \log C - bu'$ is represented by a straight line. The two next simplest cases are those where $-\log[p(u')]$ has the same convexity, up or down, over the whole range of variation of u', so that the expression $-\log p(x) - \log p(u-x)$ has an extremum for $x = u/2$. These two cases are illustrated in Figure 2.

One kind of convexity arises when $d^2 \log p(u)/du^2 \leq 0$, for all u. Then $p(u)$ decreases rapidly as $u \to \infty$, and the integrand $p(x)p(u-x)$ has a *maximum* for $x = u/2$. If that maximum is strong enough, the integral $\int_{-\infty}^{\infty} p(x)p(u-x)dx$ is likely to be made up to a great extent of the contributions of the x's in a small interval near $u/2$. Hence, a *large value of u is likely to come from two contributions u' and u'' which are almost equal.* (This is not obvious, and in particular it is not true if the concavity of $-\log p$ is turned the other way.)

Consider for example the Gaussian density

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

It is known that

$$p_2(u) = \frac{1}{\sigma\sqrt{2}\sqrt{2\pi}} \exp\left(-\frac{u^2}{4\sigma^2}\right),$$

so that the Gaussian character of density is preserved in addition, except for the value of σ. The same result can also be obtained heu-

ristically, by arguing that $p(x)p(u-x)$ stays near its maximum value $\{p(u/2)\}^2$ over some interval $D/2$ on each side of $u/2$, and is negligible elsewhere. In that case, one gets the approximate estimate:

$$p_2(u) \sim p(u/2)p(u/2)D = \frac{D}{\sigma^2 2\pi} \exp\left(-\frac{u^2}{4\sigma^2}\right).$$

In other words, the approximate estimate is correct, if one takes $D = \sigma\sqrt{\pi}$ independent of u.

Suppose now that $p(u)$ decreases slowly, so that $d^2 \log p(u)/du^2 \geq 0$, for all u. Then $p(x)p(u-x)$ has a *minimum* for $x = u/2$. It may

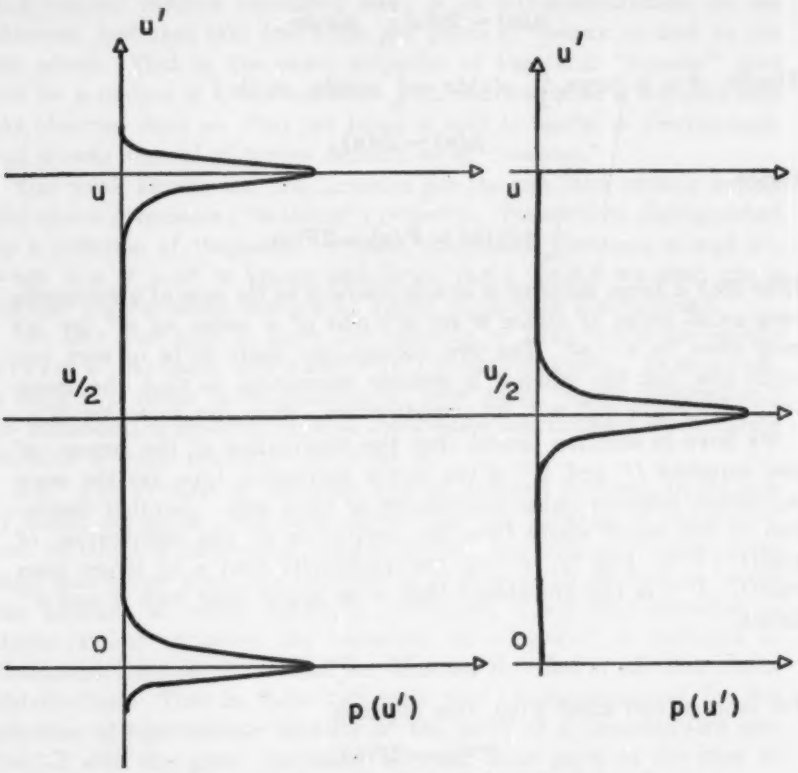

FIGURE 2: DISTRIBUTION OF U' WHEN $U' \oplus U''$ IS KNOWN

be noted that $p(x)$ must $\to 0$ as $x \to \infty$ and must either $= 0$ or $\to 0$ as $x \to -\infty$. This is compatible with a $-\log p(x)$ concave downwards, but only if the range of variation of x is truncated from below. As-

sume that u' can only be $\geq u^0$, so that its most probable value is u^0. In that case, $p(x)p(u-x)$ will have two maxima, for $x = u^0$ and for $x = u - u^0$. If they are strong enough, $p_2(u)$ will be composed mostly of the contributions of the neighborhoods of these maxima. Compare then the following two expressions:

$$p_2(u) = 2\int_0^{u/2} p(x)p(u-x)dx, \qquad \text{and} \qquad 2\int_0^{u/2} p(x)p(u)dx\,.$$

These two expressions differ only in the small contributions of x's very different from $x = 0$; that is,

$$p_2(u) \sim 2p(u)\int_0^{u/2} p(x)dx\,.$$

Finally, if u is large, $\int_0^{u/2} p(x)dx \sim \int_0^{\infty} p(x)dx$, so that

$$p_2(u) \sim 2p(u)\,.$$

Hence,

$$\int_u^{\infty} p_2(x)dx = P_2(u) \sim 2P(u)\,.$$

Note that a large value of u is now likely to be the sum of a relatively very small value of either u' (or u'') and of a value of u'' (or u') very close to $u - u^0$. The two addends are likely to be of very unequal size; but the problem is entirely symmetric, so that the mean value $E(u'/u)$ is still $u/2$, by compensation. (See also Appendix II.)

We have in addition proved that the distribution of the largest of two variables U' and U'' of the slowly decreasing type has the same asymptotic behavior as the distribution of their sum. Another derivation of this result starts from the derivation of the distribution of $\max(U', U'')$. Clearly, $P^m(u)$, the probability that u be larger than $\max(U', U'')$ is the probability that u be larger than both u' and u''. Hence,

$$1 - P^m(u) = [1 - P(u)]^2\,.$$

For large u and small $P(u)$, this becomes

$$P^m(u) \sim 2P(u)\,.$$

That is, for slowly decreasing densities, $P^m(u) \sim P_2(u)$.

A prototype of the slowly decreasing variable is the strong Pareto variable. In that case, we can write:

$$P_2(u) \sim 2P(u) = P(2^{-1/\alpha}u)\,.$$

That is, the sum of two independent and idential strong Pareto variables is a weak Pareto variable, with the same α and a scale factor u^0 multiplied by $2^{1/\alpha}$.

Likewise, any weak Pareto distribution will be invariant in addition up to the value of u^0. The proof requires an easy refinement of the previous argument, to cover the case where $-\log p(x)$ is not concave or convex all through the range of x, but $d^2 \log p(x)/dx^2$ becomes and stays negative for large values of x (which implies a quite regular behavior for $-\log p(x)$ in that region). One can show in this way that *the weak Pareto law is preserved in the addition of two (or of a few) independent random variables: there is no self-contradiction in the observed fact that this law holds for parts of income as well as for the whole.* That is, the exact definition of the term "income" may not be a matter of great concern. But, conversely, *it is unlikely that the observed data on $P(u)$ for large u will be useful in discriminating among several different definitions of "income."*

The weak Pareto and the Gaussian are the only laws strictly having the above invariance ("stability") property. They will be distinguished by a criterion of "equality" versus "inequality" between u' and u'', when $u = u' + u''$ is known and large. (2.6.) In 2.8 we shall cite a further known result concerning stable probability distributions.

We may also need to know the behavior of $p_2(u)$, when the density $p'(u')$ of U' decreases slowly and $p''(u'')$ decreases rapidly. In that case, a large u is likely to be equal to u', plus some "small fluctuation." In particular, a Gaussian error of observation concerning a weak Pareto variable is quite negligible for large u.

2.6. *The problem of addition and of division into two, in the case of stable variables.* We have shown that the behavior of the sum of two variables is determined mainly by the convexity of $-\log p(u)$: we shall later show that this criterion is in general insufficient to study the addition of *many* variables. However, if we limit ourselves to stable random variables, the convexity of $-\log p(u')$ is sufficient to distinguish between the case of the Gaussian and that of all other stable distributions. That is, these two cases may be distinguished by the criterion of approximate equality of the parts of a Gaussian sum contrasted with the great inequality between these parts in the case of all other distributions, in particular the Pareto-Lévy one.

This distribution has been used so far only to derive the distribution of $U' \oplus U''$. Suppose now that the value u of U is given and that we wish to study the distribution of u' or of $u'' = u - u'$. If the *a priori* distribution of U' is Gaussian, with mean M and variance σ^2,

then the *a priori* distribution of U is Gaussian with mean $2M$ and variance $2\sigma^2$, and the conditional distribution of u', given u, is:

$$p(u'|u) = \frac{(2\pi)^{-1/2}\sigma^{-1}\exp\left[-\frac{(u'-M)^2}{2\sigma^2}\right]\cdot(2\pi)^{-1/2}\sigma^{-1}\exp\left[-\frac{(u-u'-M)^2}{2\sigma^2}\right]}{2^{-1/2}(2\pi)^{-1/2}\sigma^{-1}\exp\left[-\frac{(u-2M)^2}{4\sigma^2}\right]}$$

$$= 2^{1/2}(2\pi)^{-1/2}\sigma^{-1}\exp\left[-\frac{(u'-u/2)^2}{\sigma^2}\right].$$

This is Gaussian, with mean value $u/2$ and variance $\sigma^2/2$. A striking feature of the result is that the *law of u' depends on u only through the mean value of U'* (see the right-hand side of Figure 2).

Let us now consider the non-Gaussian case. Insofar as our theory is adequate in income studies, the problem of division will appear in such questions as the following: if we know the sum of the agricultural and industrial incomes of an individual and if the *a priori* distributions of both these quantities are Pareto-Lévy with the same α, *then what is the distribution of the agricultural income*? This case is more involved than the Gaussian one, because we do not know any explicit analytic form for the distribution of u' or u''; we can, however, do some numerical plotting (see the left-hand side of Figure 2).

If the sum u is very large, we find that the distribution of u' has two very sharp maxima, near $u'_{\max}$ and $u - u'_{\max}$. As u decreases, the shape of this distribution of u' will change, instead of being simply translated, as in the Gaussian case. When u becomes small, more maxima may appear. They will then all merge, and the distribution of u' will differ little from that which is valid in the Gaussian case. Finally, as u becomes negative and very large, the distribution of u' will remain one with a single maximum.

Hence, bisection provides a very sharp distinction in this respect between the Gaussian and all other stable laws.

Now, consider a fairly small number N; what is the distribution of $(1/N)$-th of a stable variable? In the Gaussian case this $(1/N)$-th remains Gaussian, whatever N may be; its mean value is u/N, and its variance $(N-1)\sigma^2/N$. In the non-Gaussian case, *each part of a large u may be small or it may be close to u; most of the N parts will be small, but the largest of them will be likely to be close to the whole.*

The situation is less intuitive when N becomes very large. However, *Lévy has proved that the necessary and sufficient condition for the limit of the $\oplus$-sum $\sum U_i$ to be Gaussian is that the value u_i*

of the largest of the summands be negligible compared to the whole. On the contrary, if the limit is stable non-Gaussian and such that $E(U) = 0$, both the sum and the largest of the summands will increase roughly as $N^{1/\alpha}$, and one can even show (Darling [4]) that their ratio tends towards a limit, which is a random number having a distribution dependent upon α.

Let us now return to the discussion of 2.2. We argued there that U is the sum of N components, without knowing N. *A posteriori* this turns out to be quite acceptable, because the largest (or the few largest) of the components will cover a substantial part of the whole, essentially independent of the number N of components. This eminently desirable feature of the P–L theory is an important confirmation of its usefulness.

If, however, a P–L income is small, its components are likely to be of the same order of magnitude as in the Gaussian case. This has an important bearing upon the problem touched near the end of 2.1. Assume that one has a Census category in which income is usually rather small and which is such that when U is decomposed into parts, the sizes of the parts tend to be proportional to their *a priori* sizes. One may assimilate this behavior to that of a Gaussian unskilled worker's income (the decomposition may now refer to such things as the lengths of time during which parts of income were earned). But such behavior may also be that of a P–L variable, considered for small values of u. As a result, the seemingly fundamental problem of splitting income into two parts so that only one follows the P–L law is bound to have some solutions which are unassailable, but impossible to justify positively. Hence, it is questionable whether this problem is really fundamental.

2.7. *Addition of many weak Pareto variables; possible asymptotic invariance of the weak Pareto density if* $0 < \alpha < 2$; *non-invariance in the case* $\alpha > 2$. Applying the reasoning of 2.5 to the $\oplus$-sum $W_N = \sum_{i=1}^{N} U_i$ of N weak Pareto variables, we obtain the following value for the function P_N of the variable W_N,

$$P_N(u) \sim NP(u) \sim P(uN^{-1/\alpha}) .$$

This relationship may be expressed alternatively as follows: let a be any small probability, let $u(a)$ be the value of U such that $P[u(a)] = a$, and let $w_N(a)$ be the value of W_N such that $P_N[w_N(a)] = a$. The above approximation then becomes

$$w_N(a) \sim N^{1/\alpha} u(a) .$$

Either way, the distribution of $W_N N^{-1/\alpha}$ would be independent of

N for large values of w; in other terms, W_N would "diffuse" like $N^{1/\alpha}$. Actually, there are two obvious limitations to the validity of the approximation $P_N \sim NP$.

The first limitation is that when $\alpha > 1$ and hence $E(U) < \infty$ *this approximation can, at best, be valid if* $E(U)=0$. This gives the only possible choice of origin of U_i such that when $N \to \infty$, the distribution of $W_N N^{-1/\alpha}$ can tend to a limit, over a range of w_N such that $P_N(w_N)$ does not decrease to zero.[8]

The second limitation is that when $\alpha > 2$, the relationship $P_N \sim NP$ can at best hold in a zone of values w_N such that the total probability $P_N(W_N) \to 0$ as $N \to \infty$. This is due to the fact that U now has a finite variance $D(U)$, and it is possible to apply the classical central limit theorem to U. That is, we can now assert that, when $N \to \infty$,

$$\Pr\left\{\frac{W_N - NE(U)}{[ND(U)]^{1/2}} < x\right\} \text{ tends to } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-y^2/2)dy\,.$$

This holds over a range of values of x, of which the probability $P_N \to 1$ as $N \to \infty$. That is, an increasingly large range of values of W_N will eventually enter into the Gaussian zone, in which $W_N - NE(U)$ diffuses like $N^{1/2}$ (that is, more rapidly than $N^{1/\alpha}$). *As a counterpart, the total probability of the values of* x, *such that* $P_N \sim NP$, *must tend to zero as* $N \to \infty$.

Note that the $N^{1/2}$ diffusion is not radically changed if U is truncated to less than some fixed bound; that is, the $N^{1/2}$ diffusion represents the behavior of those values of W_N which are sums of comparable small contributions, whereas the $N^{1/\alpha}$ diffusion represents the behavior of sums of a single large term and of many very small ones.

We can now draw some conclusions concerning the relationship between the behavior of the largest of N terms U_i and the behavior of their sum. In the weak Pareto case, the two problems are identical if $N = 2$, whatever α; but as N increases they become distinct problems. The problem of the maximum is the only one which remains simple and continues to lead to a weak Pareto variable. On the contrary, the concavity of $-\log p(u')$ is not a sufficiently stringent criterion

[8] To show this, write $V = U + c$; then $V_N = U_N + Nc$ and $V_N N^{-1/\alpha} = U_N N^{-1/\alpha} + cN^{1-1/\alpha}$. When $N \to \infty$, the last term will increase without limit, so that $V_N N^{-1/\alpha}$ cannot have a non-trivial limit distribution, if $U_N N^{-1/\alpha}$ has one. Further, if $U_N N^{-1/\alpha}$ has a limit distribution. U_N/N will have the degenerate limit 0, so that we must assume that $E(U) = 0$.

If, on the contrary, $\alpha < 1$, the above argument fails, because $N^{1-1/\alpha}$ tends to zero; therefore $W_N N^{-1/\alpha}$ could have a limit distribution on a non-decreasing range of values of U, whatever the origin of U (anyway $E(U) = \infty$).

to discriminate between those cases where $P_N \sim NP$ does or does not apply, over a range of values of u having a fixed probability.

2.8. *Infinite divisibility of the stable distributions.* If U is stable, we can write, designating by $\sum$ the sums in the sense of $\oplus$,

$$U = A(N)\sum_{i=1}^{N} U_i - B(N) = \sum \{A(N)U_i - B(N)/N\} = \sum V_{Ni},$$

where the U_i are independent values of U and where $V_{Ni} = A(N)U_i - B(N)/N$. Hence, for every N, U can be considered as a sum of N independent and identically distributed variables, V_{Ni}. This is the definition of *infinite divisibility* for a random variable. Suppose that U is a P–L variable with $E(U) = 0$. Then the infinite divisibility of U is made obvious by writing the c.f. $\varphi(\zeta)$ in the following fashion:

$$\log \varphi(\zeta) = C\int_0^{\infty} (e^{i\zeta x} - 1 - i\zeta x)\,|\,d(x^{-\alpha})\,|\,.$$

To divide a P–L variable by N we need only replace C by C/N. This preserves the form of the function $\log \varphi(\zeta)$, as it should, because $(1/N)$-th of a stable variable is itself stable. If ε is chosen small enough, we may approximate $\log \varphi(\zeta)$ by the integral $\log \varphi(\zeta, \varepsilon)$, restricted to the range $(\varepsilon < x < \infty)$. Further, the term $i\zeta C\int_{\varepsilon}^{\infty} x\,|\,d(x^{-\alpha})\,|$ of $\log \varphi(\zeta, \varepsilon)$ amounts to a non-random translation of U. The essential part of $\log \varphi(\zeta)$, is, therefore, the part $C\int_{\varepsilon}^{\infty} (e^{i\zeta x} - 1)\,|\,d(x^{-\alpha})\,|$, which is a limit of approximations of the form $\sum (e^{i\zeta x} - 1)\,|\,d(x^{-\alpha})\,|$.

We may thus represent U as a limit of sums of Poisson variables. To each increment dx of the variable x there is a corresponding contribution to U equal to x multiplied by a Poisson variable of expected value $C|d(x^{-\alpha})|$. This means that a Pareto-Lévy variable may be considered as a sum of variables, each of which is closely related to the strong Pareto law. (The strong Pareto law may be truncated at any $\varepsilon > 0$, because the term $i\zeta x$ of $\log \varphi(\zeta)$ takes care of the divergence near 0 of the integral $\int_0^{\infty} (e^{i\zeta x} - 1)\,|\,d(x^{-\alpha})\,|$.)

This relationship between the strong Pareto and the P–L laws and the role of the convergence factor $i\zeta x\,|\,d(x^{-\alpha})\,|$ are made quite clear in a very intuitive physical problem of Holtsmark [9].

The problem is that of the attraction exerted on a star by an infinite uniform cloud of identical stars. Let us postpone convergence problems and consider first a very large sphere of radius R, within which N stars of unit mass are distributed at random, uniformly and independently. A final star is located at the center 0 of the sphere, and we wish to compute the *resultant of the Newtonian attractions*

exerted on 0 by the N other stars. Units will be chosen in such a way that two stars of unit mass attract each other with the force $r^{-2} = u$. Let $D = N(4R^3\pi/3)^{-1}$ be the average density of stars, and let $u^* = R^{-2}$.

Consider first the attraction of the stars located within a thin pencil (or infinitesimal cone) covering dS spherical radians, having its apex at 0, and extending in one direction only from 0. This pencil is a sum of cells, each of which is contained between some radius r and some radius $r+dr$. The volume of such a cell is $dV = dS\,d(r^3) = dS\,|d(u^{-3/2})|$. Therefore, if there were a single star in the pencil dS, the probability of its being in the volume dV would be given by the strong Pareto distribution:

$$\left|\frac{dS \cdot d(u^{-3/2})}{dS \cdot R^3}\right| = \left|d(u/u^*)^{-3/2}\right| .$$

The characteristic function of this distribution is a fairly involved function $\varphi(\zeta u^*)$. If there were N stars in the pencil, the probability that the attraction on 0 is u would have the c.f. $\varphi^N(\zeta u^*)$, which becomes increasingly more involved as $N \to \infty$.

The problem is simplified if (D remaining constant) $R \to \infty$, and $N \to \infty$. It then becomes possible to assume that the number of stars in the cell of volume dV is not fixed but is given by a Poisson random variable, with the same D as the expected density. It is clear that one can easily go from one case to the other by slightly changing the distribution of stars for large R, an area of small values of u.

In this Poisson approximation, the stars located in the volume dV will exert a total force which is a multiple of $u = r^{-2}$, the multiplier being a Poisson variable with expected value $D \cdot dS \cdot |d(u^{-3/2})|$. That is, *the total force exerted on 0 will be the sum of a number of independent discrete jumps.* (The mean relative number of jumps, with a value between some a and some b, will be $D \cdot dS \cdot (a^{-3/2} - b^{-3/2})/D \cdot dS \cdot u^{*-3/2}$, i.e., it will follow the strong Pareto law.) The c.f. of the total contribution of the pencil dS will then be approximated by the integral

$$\log \varphi_R(\zeta) = (D \cdot dS)\int_{R^{-2}}^{\infty} (e^{i\zeta u} - 1)\,|d(u^{-3/2})| .$$

The fact that the integral is extended to $u = \infty$ raises no convergence difficulty, but the careless extension of the integration down to $u = 0$ ($R = \infty$) would lead to a divergent sum. The physical reason for this is that, although each of the distant stars contributes little attraction, their number is such that their total *mean* attraction is infinite. However, one may disregard this infinite mean value, because the difference between the attraction and its mean is finite; indeed,

the fluctuation of the contribution of far away stars has the characteristic function

$$\varphi(\zeta) = \exp\left\{(D \cdot dS)\int_0^{R^{-2}} (e^{iu\zeta} - 1 - iu\zeta)\right\} \left| d(u^{-3/2}) \right| ,$$

which converges and tends to zero as $R \to \infty$. For the sake of convenience, the same correction $iu\zeta$ may also be used for all other values of u, since its effect for large u is only to add a finite translation to U. Hence, *the difference between the attraction of the stars in the pencil dS, and the mean value of this attraction, is a "positive" stable variable, or "Pareto-Lévy" variable, with* $\alpha = 3/2$.

The meaning of the stability of this attraction is that if two clouds of stars are distinguished by their colors (say red and blue) but have the same density and fill the same pencil dS, then the forces exerted on 0 by red or blue stars alone, or by both together, differ only by a scale factor and not in the analytic form of their distributions.

It is also evident why large negative values of u are unlikely compared to large positive values. A large negative u can occur only if there is an abnormally small number of stars. Moreover, the absence of any stars near 0 is a quite likely event, but alone it can at best give a bounded negative u. Therefore, a large negative u must also contain the negative contributions of stars missing far away from 0; each of these stars contributes little to U, so that the number of missing stars must be very large, and this is very unlikely.

On the contrary, an unboundedly large positive u may be obtained from the presence of a *single* star near 0, *irrespective of the density of stars elsewhere*; such an event is far more likely than the combination of events required for a negative u. It is easy to check the fact that the distribution of U has the same asymptotic behavior for large u as the distribution of the attraction of the nearest star.

3. CONCLUSION

It is to be hoped that this first application of the stable laws will stimulate interest in a more detailed study of their properties. Most of the usual procedures of statistics must be revised because of the infinite variance of U; such new problems arise, as that of the best choice of the origin u'', in such fashion that the P–L density $p(u)$ and the strong Pareto density $[(u - u'')/u^0]^{-\alpha}$ coincide over as large a range of values of u as possible.

Another series of problems concerns the comparison of the P–L curve with the empirical data in the region of intermediate values of u.

A final problem concerns the sign of $\alpha - 2$. We have referred to it several times, but a further discussion can be pursued only within the framework of a theory of P–L processes. (One indication may already be found in Section 2 of [19].) To settle this problem, it will probably be necessary to introduce some dependence between the additive components of U; this must be done carefully, however, if one wants to avoid obtaining a wholly indeterminate answer. See also footnote 3.

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APPENDIX I

The behavior of the P–L density for large negative arguments is not classical. To derive it, we note first that the form of the bilateral generating function $G(b)$ (which is not commonly used) may be deduced from the commonly used characteristic function, with the help of some standard theorems on Fourier transforms in the complex plane. From the existence of $G(b)$ it follows that $p(u)$ must decrease faster than any form $\exp(-|bu|)$, when $u \to -\infty$.

To show exactly how fast $p(u)$ decreases, write for convenience $v = -u$ and $f(v) = -\log p(v)$; $f(v)$ will increase faster than any linear form of v. Write further:

$$G(b) = \int_{-\infty}^{\infty} \exp(bv)p(v)dv = \int_{-\infty}^{\infty} \exp[bv - f(v)]dv = \int_{-\infty}^{\infty} \exp[h(v)]dv .$$

If b is large, the integrand $\exp(h)$ must have a maximum for the value of v such that $b = f'(w)$. Near that point, we can write:

$$h(v) = [bw - f'(w)] - (1/2)(v - w)^2 f''(w) + (1/6)(v - w)^3 f^{(3)}(w) + \cdots .$$

If we could limit oneself to terms of order 0 and 2, we would find for $G(b)$ the approximation $\exp[bw - f'(w)][2\pi/f''(w)]^{-1/2}$. Let us investigate the possibility of equating this approximation to $\exp(b^\alpha)$, with a $f(v)$ of the form Kv^c; if we take account of the term $\exp[bw - f'(w)]$ alone, the approximation may indeed be made equal to $\exp(b^\alpha)$ by taking $c = \alpha(\alpha - 1)^{-1}$. The term $[2\pi/f''(w)]^{-1/2}$ weakens the result somewhat, since, instead of having $\log[-\log p(v)] = \log K + c \log v$, we can only assert that $\log(-\log p)/\log v \to c$, as $v \to \infty$. Finally, consider the terms other than those of orders 0 and 2: we note that the term in $(v - w)^3$ gives a non-negligible contribution to $G(b)$, only as long as $(v - w)$ is of the order of magnitude of $[f''(w)]^{-1/2} \sim w^{1-\alpha[2(\alpha-1)]^{-1}}$. Then the term in $(v - w)^3$ is of the order of magnitude of $w^{-\alpha[2(\alpha-1)]^{-1}}$ and is negligible; similarly the terms of higher order do not modify the behavior of $p(v)$.

APPENDIX II

We can exhibit the invariance of the weak Pareto variables in a specific example. Let $0 < \alpha < 1$, and consider the discrete variable having the following $Z(b)$ as discrete (one-sided) generating function:

$$Z(b) = \sum_{n=0}^{\infty} \exp(-bn)p(n) = 1 - C(1 - e^{-b})^\alpha \qquad \text{(where } 0 < C < 1\text{)} .$$

All $p(n)$ are positive and less than 1, and $\sum_{n=1}^{\infty} p(n) = 1$. For large n,

$$p(n) \sim \frac{Cn^{-(\alpha+1)}}{\Gamma(-\alpha)} .$$

The sum of two variables of this type has the generating function:

$$Z_2(b) = Z^2(b) - 1 - 2C(1 - e^{-b})^\alpha + C^2(1 - e^{-b})^{2\alpha} ,$$

so that

$$P_2(n) \sim \frac{2Cn^{-(\alpha+1)}}{\Gamma(-\alpha)} + \frac{C^2 n^{-(2\alpha+1)}}{\Gamma(-2\alpha)} = 2p(n) + \text{correction} .$$

For large n, the second part of the right-hand term becomes negligible compared to the first. If $\alpha = 1/2$, it is zero, so that the range of values of n in which it may be neglected increases as α tends to 1/2.

If $1 < \alpha < 2$, we must add a factor in $(1 - e^{-b})$ to obtain an acceptable generating function; consider for example:

$$Z(b) = 1 - C(1 - e^{-b}) + C'(1 - e^{-b})^\alpha$$

where $0 < \alpha C' \leq C \leq C' + 1$, so that $C' \leq (\alpha - 1)^{-1}$.

Now

$$p_2(n) \sim \frac{2Cn^{-(\alpha+1)}}{\Gamma(-\alpha)} + \frac{2CC'(\alpha+1)n^{-(\alpha+1)}}{\Gamma(-\alpha)} + \frac{C'^2 n^{-(2\alpha+1)}}{\Gamma(-2\alpha)} .$$

For large n, the second and third terms become negligible for all α. The ratio of the coefficients of the first and of the second term depends little upon α; but the ratio of the coefficients of the third and first terms is ruled by $\Gamma(-\alpha)/\Gamma(-2\alpha)$, which is zero for $\alpha = 3/2$, but may become large elsewhere. As a result, the range of values of n over which the third term is important may be large.

Each time α increases past an integral value, the sign of $C(1 - e^{-b})$ must be changed, and another polynomial term must be added to $Z(b)$, to keep it a generating function. The number of corrective terms of $p_2(n) - 2p(n)$ increases, as well as the range of values of n in which the corrective terms are appreciable.

Similarly, as more than two terms are added, $p_N(n) - Np(n)$ is vitiated over an increasing range of values of n. Let $N \to \infty$, and observe the weighted sums of the variables U_i, whose values are the integers n. ($\sum$ remains a summation in the sense of $\oplus$.)

For $0 < \alpha < 1$, it is sufficient to consider the expression $W_N = N^{-1/\alpha} \sum U_i$, since its g.f. is $Z^N(N^{-1/\alpha}b)$, which tends to $\exp(-Cb^\alpha)$ when $N \to \infty$, as it should.

If $1 < \alpha < 2$, one must consider the expression $W_N = N^{-1/\alpha} \sum (U_i - M)$, where M, the mean value of U_i, is easily found to be C. The g.f. of W_N is clearly $\exp(NCb)Z^N(N^{-1/\alpha}b)$ and when $N \to \infty$, it tends to $\exp(Cb^\alpha)$, as it should. It is easily seen that if we choose for M a value different from C, the g.f. of W_N will not tend to a non-degenerate expression.

If the value of α is higher, no linear renorming of U_i can eliminate from $\log[Z(b)]$ the square term Kb^2, and hence, the best normed sum of the U_i is the classically normed $N^{-1/2} \sum (U_i - M)$, which tends to a Gaussian, whatever the value of α.

APPENDIX III

This appendix concerns the transformation $V = \log U$, which is used in most classical theories of income distribution.

The strong objections against this transformation do not apply at all in the case of a law due to Estoup and Zipf, which is formally similar to that of Pareto, but relates to word

frequencies, and which we have studied since 1951 (cf. [14], [16]). In the formal expression of that law U is replaced by the "rank" of a word, where words are ordered by decreasing frequencies. Hence $\log U$ has an intrinsic meaning as "cost of coding"; in particular, the addition of "costs" is quite meaningful.

In the case of income, the transformation $V=\log U$ is justified in our [18] and [19], and in a more detailed article which ought to appear soon.

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INTERNATIONAL ECONOMIC REVIEW
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ON EQUILIBRIUM GROWTH OF CAPITAL AND LABOR*

By YŌICHI SHINKAI[1]

IT IS one of the characteristic conclusions of the Harrod-Domar model that it is impossible to sustain full employment unless the warranted rate of growth coincides with the natural rate, and the initial stock of capital just equals the initial supply of labor multiplied by "the" capital-labor ratio. This "knife-edge notion of unstable balance" follows from the crucial assumption that production takes place under conditions of fixed proportions of capital and labor. Thus Professor Solow [5] has shown that when production takes place under the neo-classical conditions of variable proportions and constant returns to scale, equilibrium growth is very likely to be stable.

His conclusion is based on a one-sector analysis. In this paper we present a two-sector model, where all industries are divided into two branches, capital-good and consumption-good industries. On the assumption that only one technique of production is available to each of the two sectors, we find that if the capital-good industry is more capital-intensive than the consumption-good industry, the balanced growth of capital and labor is necessarily unstable; if the former is more labor-intensive than the latter, the balanced growth will be stable. Thus balanced (or equilibrium) growth in our model is of the Harrod-Domar type or of the Solow type, depending on the relative magnitudes of the capital-labor ratios of the two sectors.

A further historical note may be added. As the reader will find, our model has much in common with Mrs. Robinson's "simple model" of capital accumulation [4, pp. 63–100] and Professor Morishima's interpretation of her model [3], and our equilibrium growth corresponds to her "golden age" (under the hypothesis of no technical progress). If our stability condition is satisfied, full utilization of capital and full employment of labor can proceed harmoniously. If the "golden-age" path is unstable, on the other hand, the maintenance of full employment will, in most cases, lead to a breakdown of capital accumulation.

1. ASSUMPTIONS AND MODEL

We start with the following assumptions:

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[1] Thanks are due to Professors L. R. Klein, M. Morishima, and the referee of this paper for many helpful comments.

(1) There are only two kinds of goods, capital goods and consumption goods. Capital goods are not demanded by consumers, and consumption goods are not used for production.

(2) Each good requires for its production capital goods and labor as inputs; we abstract from the existence of land and scarce natural resources.

(3) Only one technique is available to each industry, i.e., all technological coefficients are constant.

(4) The period of production in each sector is ignored.

(5) Capital stock is freely transferable from one sector to the other.

(6) The problem of depreciation is ignored, so that the output of the capital-good industry is defined net of the wear and tear of the capital stock of the whole economy. (This means that the output of the capital-good industry is equal to the rate of net investment of the economy.)

(7) Labor is homogeneous; the wage rate is always the same for all workers.

(8) The supply of labor grows at a given constant rate.

(9) Workers only consume; capitalists only save; all savings are reinvested by capitalists.

The notation used is as follows:

Sector 1 ... capital-good industry (or investment sector),
Sector 2 ... consumption-good industry (or consumption sector),
X_i ..... output of sector i,
N_i ..... labor employed in sector i,
a_i ..... capital-labor ratio of sector i,
b_i ..... labor-input coefficient of sector i,
W ..... real wage rate,
N ..... total supply of labor,
K ..... stock of capital in existence,
n ..... growth rate of N.

The conditions for equilibrium (no excess capacity, no excess demand for consumption goods and full employment of labor) may be written as follows:

$$K = a_1N_1 + a_2N_2 , \tag{1}$$

$$X_2 = W(N_1 + N_2) , \tag{2}$$

$$N = N_1 + N_2 . \tag{3}$$

Given the stock of capital in existence K and the available supply of labor N, the equations (1), (2) and (3) will determine three variables N_1, N_2 and W, because we can reduce X_2 to N_2 by means of

Assumption 3. Once N_1 and N_2 are determined, the outputs of both sectors, X_1 and X_2 are also determined. This is what happens in a single period.

In the next period the stock of capital in existence is increased by X_1, while the labor force is larger by nN than before. With these new values of K and N inserted, (1), (2) and (3) will give new values of N_1, N_2 and W, of which the first, in turn, will determine the rate of investment, X_1, in the second period. Thus the accumulation of capital continues, always satisfying the conditions for equilibrium and full employment.

This dynamic aspect of our model may be analysed by taking

$$\dot{K} = X_1 \tag{4}$$

into account explicitly. Remembering that the growth rate of N is n, we easily obtain from (1), (3) and (4):

$$\begin{aligned} a_1\dot{N}_1 + a_2\dot{N}_2 &= mN_1 , \\ \dot{N}_1 + \dot{N}_2 &= nN_1 + nN_2 , \end{aligned} \tag{5}$$

where m stands for the output per man in the investment sector (i. e., the inverse of b_1). In order for these equations to have solutions, the determinant $\begin{vmatrix} a_1 & a_2 \\ 1 & 1 \end{vmatrix}$ must not vanish, that is:

$$a \equiv a_1 - a_2 \neq 0.^2$$

This same condition must also hold for the existence of solutions to the static system (1) – (3) above.

Now let λ_1 and λ_2 be the roots of the characteristic equation:

$$\begin{vmatrix} m - a_1\lambda & -a_2\lambda \\ n - \lambda & n - \lambda \end{vmatrix} = 0 . \tag{6}$$

Then the general solution of the equations (5) may be written in the form:

$$\begin{aligned} N_1 &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} , \\ N_2 &= Ce^{\lambda_1 t} + De^{\lambda_2 t} , \end{aligned} \tag{7}$$

where A, B, C and D depend on the initial conditions and the coefficients of the equations. By writing the initial stock of capital as K_0 and the initial supply of labor as N_0, we have:

[2] It is apparent that if $a = 0$, i. e., if the capital-labor ratios of both sectors are equal, our model loses most of the characteristics of a two-sector model.

$$(8)\qquad \begin{cases} A = cN_0\,, \\ B = (K_0/c - N_0/b_1 n)d\,, \\ C = (1-c)N_0\,, \\ D = (N_0/b_1 n - K_0/c)d = -B\,, \end{cases}$$

where:

$$c = a_2 b_1 n/(1 - ab_1 n)\,,$$
$$d = c/a\,.$$

Solving (6), we find:

$$\lambda_1 = n \quad \text{and} \quad \lambda_2 = 1/ab_1\,.$$

2. STABILITY OF EQUILIBRIUM GROWTH

In normal circumstances the capital coefficient, $a_1 b_1$, of the investment sector will be at most 10, and the growth rate of labor, n, can hardly be greater than 0.02, so that we may safely assume:[3]

$$1 - a_1 b_1 n > 0\,.$$

Hence $0 < c < 1$ in (8), which means that both A and C in (7) are strictly positive.

First, we shall be concerned with the case where the initial stock of capital, K_0, is equal to $cN_0/b_1 n$. It follows from (8) that B and D vanish in (7). Thus we obtain:

$$(9)\qquad N_1 = Ae^{nt} \quad \text{and} \quad N_2 = Ce^{nt}\,.$$

By virtue of the assumption of rigid production coefficients, X_1, X_2, K_1 and K_2 grow at the common rate n. In this case the economy is in a state of equilibrium (or balanced) growth.

Second, let K_0 be smaller than $cN_0/b_1 n$, a situation which may be referred to as the case of capital shortage.[4] If, in this case, the consumption sector is more capital-intensive than the investment sector, i.e., if a $(= a_1 - a_2)$ is negative, then d is negative, so that $B > 0$ and $D < 0$ in (7). Clearly N_1 is larger than in equilibrium growth, and N_2 is smaller. However, since λ_2 is negative in this case, N_1 and N_2 eventually approach the equilibrium growth path. In the earlier stages capital is accumulating faster than the available labor increases, and capital stock will catch up with labor in the end.[5]

[3] It should be noted that although each of the values, $a_1 b_1$ and n, taken separately depends on the length of period we choose, the product of the two is independent of it.

[4] We assume that $K_0 - cN_0/b_1 n$ is not so small as to make N_1 or N_2 in (7) negative.

[5] It is interesting to note that some economists seem to regard Mrs. Robinson's model as stable. Thus Professor Champernowne [1] and Dr. Little [2] present models of which the capital-labor ratio of the investment sector is zero.

If the investment sector is more capital-intensive than the consumption sector, on the other hand, i.e., if a is positive, we find that $B < 0$ and $D > 0$, and that N_1 is smaller and N_2 is larger than the equilibrium values (9). The output of capital goods, therefore, is smaller than in equilibrium growth; this will aggravate the shortage of capital in the next period. In fact, if $a > 0$, then $\lambda_2 > 0$, so that $Be^{\lambda_2 t}$ gets smaller (algebraically) and $De^{\lambda_2 t}$ gets larger, when t increases. Furthermore, since $a_1 b_1 n$ is assumed to be less than unity, it can easily be shown that if $a > 0$, $\lambda_2 > \lambda_1$. Hence the term $Be^{\lambda_2 t}$ will eventually become dominant in the solution (7), so that N_1 will ultimately become zero. When N_1 is zero, the output of capital goods ceases, and capital accumulation will be at a standstill.

Finally, much the same argument can be applied to the case of capital surplus ($K_0 > cN_0/b_1 n$). In this case, too, equilibrium growth is stable if a is negative and unstable if it is positive. The main difference from the case of capital shortage lies in the way in which capital accumulation comes to a standstill when equilibrium growth is unstable. If $K_0 > cN_0/b_1 n$ and $a > 0$, then $B > 0$, $D < 0$, and $\lambda_2 > \lambda_1 > 0$ in (7). As time goes by, N_1 becomes ever larger and N_2 smaller than the equilibrium values (9). This state of affairs must come to an end when N_2 is zero, because in that event the output of consumers' goods will be nil.

Thus if, equilibrium growth is unstable, the slightest disturbance of initial conditions away from the path of equilibrium growth will sooner or later result in a state where either N_1 or N_2 is zero. After this point of time, it is impossible to maintain full utilization of capital and full employment of labor simultaneously. Unemployment of labor or excess capacity is inevitable according as the initial stock of capital K_0 is smaller or greater than $cN_0/b_1 n$.

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A THEORY OF INTERSTATE DIFFERENCES IN THE RATES OF GROWTH OF MANUFACTURING EMPLOYMENT IN A FREE MARKET AREA*

BY JEROME L. STEIN[1]

IN THIS ARTICLE we shall develop a theory to explain the existence of interstate differences in rates of growth of manufacturing employment among states linked together in a free market area. This free market area is characterized by free trade, free capital movements, and the use of a common currency. The contemporary theories of economic growth have concerned themselves with a closed economy and have not explained the existence of interstate differences in growth rates in a free market area.

The United States of America constitute a free market area and, as such, provide an excellent subject for analysis. Over the period covered, certain empirical regularities have been discovered concerning the process of growth. These regularities are described in Part 1 and are the phenomena to be explained by a theory of growth. The need for a theory of growth is developed in Part 2, where the inadequacy of simplistic rationalizations is demonstrated. Part 3 presents a theory of growth which implies the phenomena cited in Part 1. The explanatory power of this theory is explicitly stressed in Part 4.

There exist an unlimited number of theories which are compatible with a given set of data. The virtues of the theory presented here are: (a) it is testable against the available data; (b) it implies the data cited in Part 1; (c) the theory is conceptually simple and is based upon a competitive model; and (d) the crucial variable is the rate of growth of the labor supply curve rather than a nebulous variable like natural resources endowment.

1. THE EMPIRICAL PHENOMENA TO BE EXPLAINED

A. *The Stability of Relative Growth Rates Within the United States.*

The first phenomenon to be explained is that interstate differences in rates of growth of production worker employment display certain

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patterns of regularity. Certain states have *consistently* grown more rapidly than the national average; other states have *consistently* grown at rates below the national average.

We have data on the average number of wage earners in manufacturing establishments for forty-five states over the eight decades from 1869 to 1949. Define as a "success" the event that a given state exhibits a greater rate of growth of wage earners' employment than the national average over a given decade. From 1869 to 1949 a state can have no fewer than zero and no more than eight successes. The distribution of states by number of successes is given in columns 1 and 2 of Table 1 below. There are three states for which there are no data on the early periods. This leaves forty-five states with eight observations each.

We must show that the distribution of states by number of successes is not a phenomenon produced by chance. There were a total of 184 successes out of 360 ($= 45 \times 8$) observations. Assume that the event of a success or a failure is a Bernoulli trial with a fixed probability of one-half. The distribution of states by successes would then be the same as the distribution of forty-five families, with eight children each, by the number of boys per family. The probability of k successes in eight trials would be $b(k; 8, \frac{1}{2})$ or $\binom{8}{k}(\frac{1}{2})^8$. The *expected* number of states would be equal to 45 times $b(k; 8, \frac{1}{2})$. This is given in column 3 of Table 1. The data were grouped to insure at least five observations per cell.

TABLE 1

THE DISTRIBUTION OF STATES BY NUMBER OF SUCCESSES, 1869–1949

Number of Successes	Number of States	Expected Number of States
0	3	
1	3 } 10	6.5
2	4	
3	6	10
4	8	12
5	8	10
6	9	
7	3 } 13	6.5
8	1	
	45	45

Source: R. A. Easterlin, "Estimates of Manufacturing Activity," in E. S. Lee, *et. al.*, *Population Redistribution and Economic Growth, United States, 1870–1950* (Philadelphia: The American Philosophical Society, 1957).

A *chi*-square test produces a value of *chi*-square of 11.7 with four degrees of freedom. This is statistically significant at the two per cent level. We may therefore reject the hypothesis that the data in columns 1 and 2 were produced by Bernoulli trials with a probability of one-half.

Certain states consistently have grown more rapidly than the national average; other states consistently have grown less rapidly than the national average. A satisfactory theory of growth must explain this phenomenon.

B. *Variables Which are Significantly Associated with the Rate of Growth of Manufacturing Production Worker Employment.*

Let us focus upon a period between two business cycle peaks, 1919 to 1953. During this period, the rate of growth of manufacturing employment in a state (i) was negatively related to the ratio of manufacturing to total state employment, (ii) was positively related to the rate of in-migration into a state, and (iii) was not significantly related to the level of the wage prevailing in the state's manufacturing industries.

1. *The aggregate approach.*

The rate of growth of manufacturing production worker employment between 1919 and 1953 was computed from the *Census of Manufactures* and the *Annual Survey of Manufactures*. Call this variable Y. The ratio of manufacturing to total state employment, (ratio X_1), was computed for 1940, a census year between the initial and terminal dates.

The rate of (in)migration by state was calculated over the period 1920–50. We summed the total migration into the state of people in the age group 15–64 during the period 1920–50. This figure was divided by the total number of people in the age group 15–64 who lived in the state during the period 1920–50.[2] Call this resulting variable X_2. Given the values of X_1 and X_2 by state, how significant are they in explaining growth rates of employment by states from 1919 to 1953?

X_1 and X_2 are expressed in standard measure and become x_1' and x_2' respectively.[3] Y becomes y when expressed in standard measure and

[2] Everett S. Lee, "Migration Estimates," in Everett S. Lee, Ann R. Miller, Carol P. Brainerd, and Richard A. Easterlin, *Population Redistribution and Economic Growth, United States, 1870–1950* (Philadelphia: The American Philosophical Society, 1957), Table P–1.

[3] X_1 is the ratio of manufacturing to total state employment multiplied by 1,000. Hence, 0.2135 became 213.5 which was rounded to 214. In standard measure $x_1' = (X_1 - \overline{X}_1)/S_1$. X_2 is the ratio of in-migration to state population of people of labor force age plus 0.184 to eliminate negative numbers. The resulting figure was multiplied by 1,000. Hence, −0.106 becomes 78. $x_2' = (X_2 - \overline{X}_2)/S_2$.

$\hat{y}$ is the value of y predicted from the regression equation. When the variables are expressed in standard measure, the regression coefficients of x_1' and x_2' have the same standard error. A double star indicates that the regression coefficient is significantly different from zero at the one per cent level.

(A) $$\hat{y} = \underset{**}{-0.5942}x_1' + \underset{**}{0.3912}x_2', \qquad R = 0.626^{**}.$$

The standard error of the regression coefficients is 0.12. We conclude that:

(i) we must reject the null hypothesis that the ratio of manufacturing to total state employment has no effect upon the growth rate. The regression coefficient of x_1' is -0.5942 and is significantly different from zero at the one per cent level; and

(ii) we must also reject the null hypothesis that the rate of migration has no effect upon the growth rate. The regression coefficient of x_2' is 0.3912, which is statistically significant at the one per cent level.

If we try to predict growth rates by substituting the 1939 level of the production workers' annual earnings[4] for the migration rates, we do not meet with success. It is true that migration occurs from low- to high-wage states, but the level of the wage has an ambiguous net effect upon the growth rate. Call the average annual earnings of a production worker in manufacturing X_3. Let X_1 and Y have the same meanings as above.

(B) $$\hat{y} = \underset{**}{-0.5099}x_1' + 0.1025x_3', \qquad R = 0.509^{**}.$$

The standard error of the regression coefficient is 0.1290. We cannot reject the null hypothesis that the level of the wage has no effect upon the rate of growth. The regression coefficient 0.1025 is not significantly different from zero at the five per cent level.

2. *An industry by state analysis.*

Interstate differences in industrial composition were ignored in the immediately preceding section. Insofar as interstate differences in industrial composition exist, it is possible that employment in state A grows more rapidly than it does in state B; nevertheless, for each industry which is located in both states, employment grows more rapidly in state B. That is, state A may have a concentration of rapidly growing national industries, whereas state B may have a concentration of slowly growing national industries.

When interstate differences in industrial composition are taken into account, are the variables X_1 and X_2 still significantly related to the

[4] The year 1939 was a census year and is close to the year in which X_1 was measured.

state's rate of growth? A hypothesis, subject to test, can be formulated as follows:

The probability that manufacturing industry X will grow more rapidly (in terms of production worker employment) in state A than in the United States as a whole depends: (a) negatively upon the ratio of manufacturing to total employment in the state; and (b) positively upon the rate of migration into the state.

Fourteen manufacturing industries and forty-eight states were examined between 1947 and 1953.

The industries were:

SIC Number	*Industry*
22	Textile Mill Products
23	Apparel and Related Products
25	Furniture and Fixtures
26	Paper and Allied Products
27	Printing and Publishing Industries
28	Chemicals and Allied Products
30	Rubber Products
31	Leather and Leather Products
33	Primary Metal Industries
34	Fabricated Metal Products
35	Machinery (except Electrical)
36	Electrical Machinery
37	Transportation Equipment
38	Instruments and Related Products.

Two industries were excluded because they are market oriented, and two industries were excluded because they are natural resource oriented. In the first category we have SIC Number 20, Food and Kindred Products, and SIC Number 28, Chemicals. The heavy industrial chemicals such as sulfuric acid cannot be transported very far. There are also many chemical products, such as the by-products of refineries, which are natural resource oriented. In the second category we have SIC Number 21, Tobacco Products, and SIC Number 24 Lumber and Wood Products (except furniture), which must be located near forests.

The rate of growth of production worker employment in each industry, in a given state, was compared with the rate of growth of production worker employment in the same industry in the United States as a whole. If the industry grew more rapidly (in terms of production worker employment) in a given state than it did in the United States, we score a success. Otherwise, we score a failure.

The forty-eight states were ranked in terms of the ratio of manufacturing to state employment in 1950. Row 1 contains the top third of manufacturing states, row 2 contains the middle third of manufacturing states, and row 3 contains the bottom third. In column 1 we place the states[5] which had net in-migration of civilian population from 1945 to 1954; in column 2 we place the states which had net out-migration of civilian population during this period. The following 3×2 table was then obtained. It is important to note that labor migrates from non-manufacturing to manufacturing states.[6] Hence, extreme caution must be used in interpreting this table.

On the whole 281 industries and states were covered, of which 130 grew more rapidly than their national counterparts. Thus, the probability of success in the entire sample is 0.4626. The hypothesis subject to test claims that the probability of success is negatively related to the ratio of manufacturing to total employment. As we descend from row 1 to row 2 to row 3, the probability of success is expected to increase.

This, in fact, occurs, whether we look at the states with net in-migration or with net out-migration. Among the states which had net in-migration, the probability of success increased from 0.2824 to 0.6071 to 0.8571 as the ratio of manufacturing to total state employment decreased. Similarly, among the states which experienced net out-migration, the probability of success increased from 0.50 to 0.5769 to 0.7857 as the ratio of manufacturing to total state employment decreased.

A test of statistical significance was applied to the rows. The probability of success in the entire sample is 0.4626. We may ask whether the differences in probabilities of success among the rows arise solely from random sampling out of a common population with a probability of success of 0.4626.

Row 1 contains 147 observations; row 2 contains 106 observations; and row 3 contains 28 observations. If the rows merely represent random samples from a population with a probability of success of 0.4626, then we should expect 68, 49, and 13 successes in rows 1, 2, and 3 respectively. *Chi*-square is 18.92** with two degrees of freedom. This is significantly different from zero at the one per cent level. We may reject the hypothesis that rows 1, 2, and 3 are random samples drawn from a common population.

[5] Several states were not included in the table below due to the paucity of data, resulting from the disclosure rule.

[6] In regression (A) above r_{12} was positive though not significant at the five per cent level.

TABLE 2

PROBABILITIES OF GROWING MORE RAPIDLY THAN THE NATIONAL AVERAGE, 1947–53*

	Net In-migration	n	s	Net Out-migration	n	s	
Top third in terms of ratio of manufacturing to total employment	New Hampshire	3	1	Maine	4	1	
	Massachusetts	13	4	Rhode Island	5	2	
	Connecticut	11	2	North Carolina	4	3	
	New York	14	4	South Carolina	3	2	
	New Jersey	13	5		16	8	$n = 147$
	Pennsylvania	14	4	$p_{12} = 0.5000$			$s = 45$
	Ohio	13	4				$p = 0.3061$
	Maryland	9	4				
	Illinois	12	0				
	Michigan	12	5				
	Wisconsin	12	2				
	Delaware	5	2				
		131	37				
	$p_{11} = 0.2824$						
Middle third	Minnesota	4	1	Vermont	2	1	
	Maryland	11	5	Iowa	6	4	
	Oregon	4	2	Missouri	10	5	
	California	9	9	Virginia	9	5	
		28	17	West Virginia	5	1	$n = 106$
	$p_{21} = 0.6071$			Georgia	7	4	$s = 62$
				Kentucky	7	4	$p = 0.5849$
				Tennessee	9	4	
				Alabama	6	4	
				Arkansas	6	5	
				Louisiana	6	4	
				Washington	5	4	
					78	45	
				$p_{22} = 0.5769$			
Bottom third	Florida	2	2	Nebraska	2	0	
	Texas	10	9	Kansas	4	4	
	Colorado	1	0	Mississippi	3	3	$n = 28$
	Arizona	1	1	Oklahoma	3	2	$s = 23$
		14	12	Utah	2	2	$p = 0.8214$
	$p_{31} = 0.8571$				14	11	
				$p_{32} = 0.7857$			
	$n = 173$			$n = 108$			$n = 281$
	$s = 66$			$s = 64$			$s = 130$
	$p = 0.3815$			$p = 0.5926$			$p = 0.4626$

Source: For growth data, *Census of Manufactures, 1947* and *Annual Survey of Manufactures, 1953*. For data on the ratio of manufacturing to total employment, *Statistical Abstract, 1954*. For migration, U. S. Bureau of the Census, *Current Population Reports, Population Estimates*, Series P-25, No. 124, Oct. 24, 1955, p. 7.

* n = number of cases, s = number of successes.

The hypothesis also claims that the probability of success is positively related to the rate of migration. States having net in-migration are expected to have a higher probability of success than states having net out-migration. The data presented in Table 1 are not conclusive, because the rate of migration and the ratio of manufacturing to total employment are positively correlated. Except for row 1, the

probability of success is greater in column 1 than in column 2. Given the ratio of manufacturing employment, states receiving net in-migration did better than states having net out-migration in two out of three cases. The states in row 1, column 1 are heavily industrialized compared to North Carolina, South Carolina, and Maine in row 1, column 2. Hence, the poorer performance of states in row 1, column 1 compared to the states in row 1, column 2 may merely reflect the dominant effect of industrialization (i.e., the ratio of employment in manufacturing) in inhibiting growth. The multiple regression analysis used in Section 1 above is a finer test of the hypothesis than the probability test used here, because of the correlation between migration and industrialization.

2. A REJECTION OF SIMPLISTIC RATIONALIZATIONS

It would seem that the above empirical findings could be rationalized with a little commonsense reasoning and a little economic theory.

(A) *Simplistic view*: Of course, states with low proportions in manufacturing grow fast in manufacturing employment. Those are the states where there is room for expansion: the percentage growth from a small base looks big.

Manufacturing employment grew from M_0 to M_1, yielding a growth rate of $\Delta M/M_0$. If ΔM were randomly distributed among states, then it is clear that $\Delta M/M_0$ would be greater among states with low M_0 than among states with high M_0. This is the only rationale I know for the argument that high growth rates are to be expected from states with low bases. It was observed that $\Delta M/M_0$ was negatively associated with M_1/L_1 (the ratio in manufacturing at an intermediate period). Hence, the simplistic view maintains that M_1/L_1 is positively correlated with M_0. (L_1 is total state employment.)

I tested the simplistic hypothesis directly: namely, that the differences in growth rates are attributable to variations in M_0. A statistical test was made of the null hypothesis: the initial values of M_0 are not different between states with high $\Delta M/M_0$ and states with low $\Delta M/M_0$. This test attempts to determine whether the high growth states started from lower bases than did the low growth states.

The states were divided into two groups, the states which had above median rates of growth of manufacturing production worker employment from 1919 to 1953, and those which did not. The initial amounts of manufacturing production worker employment were listed by state, and a test for the difference between two means was performed. To be sure, the mean manufacturing production worker employment in

1919 was lower among the high growth states than among the low growth states, but this difference is not significant at the five per cent level. Hence, we may not reject the null hypothesis that the 1919 levels of employment in the two sets of states were drawn from the same population. This effectively rejects the argument that the differential growth rates were produced merely by the different bases.

(B) *Simplistic view*: One would expect growth and in-migration to be positively associated. After all, in-migration is evidence, in itself, of good manufacturing job opportunities. People go to the jobs; hence, employment growth looks biggest where people are going in largest numbers.

The surprising thing is that the rate of in-migration (X_2) and the rate of growth (Y) are *not* correlated at the zero order. The total correlation between X_2 and Y is 0.2464, which is not significantly different from zero at the five per cent level. It is by no means obvious that the rate of migration is positively associated with the rate of growth.

Only in a partial correlation does the influence of migration upon growth manifest itself. The regression coefficient of x_2' (in regression (A), above) is statistically significant at the one per cent level. When the influence of X_1 (the ratio of manufacturing employment) is taken account of, then the rate of migration and the rate of growth are positively and significantly associated.

One must explain why the total correlation $r(Y, X_2)$ is not significant whereas the partial correlation $r(Y, X_2 \cdot X_1)$ is significant. The explanation must be based upon economics and not solely upon the truism that (X_1, X_2) are correlated.

Moreover, one must explain why the wage level (X_3) is not significantly associated with the rate of growth (Y). A simplistic theory fails to do these things.

3. A THEORY OF GROWTH

The inability of the simplistic rationalizations to explain the above phenomena necessitates the construction of a theory of growth which can explain the empirical regularities described in Part 1 above.

Every theory is a simplified description of reality. Assumptions are made which may not be exact descriptions of reality, in order to expose the basic forces which operate to produce the phenomena to be explained. The introduction of "more realistic" assumptions may merely complicate the analysis and prevent us from understanding the nature of the forces at work.[7] The art of theorizing consists of find-

[7] See Milton Friedman, *Essays in Positive Economics* (Chicago, 1953), ch. I.

ing the crucial variables that produce the phenomena to be explained and not getting involved with a myriad of variables which are of secondary importance. The crucial tests of a theory are (i) whether its predictions are compatible with the data to be explained and (ii) whether it helps us to understand the workings of our economic system. The theory developed here should be evaluated in terms of its ability to explain[8] the material in Part 1 above, and not in terms of the verisimilitude of the assumptions made.[9]

A. *Assumptions.*

1. Assume that the f.o.b. price of a product produced by a given manufacturing industry is the same for each firm, regardless of its regional location. Transport cost varies directly with the distance of the consumer from the factory concerned. The transport cost per mile may differ among firms. Consumers are assumed to buy from the firm for which the total price is cheapest. As Schneider[10] has shown, well defined market areas exist for the various firms, under these assumptions.

Moreover, let us assume that the producers in any single state do not significantly affect the price of a given product by varying the state's rate of output. This assumption is convenient, but not crucial, for analysis. Let the f.o.b. price of output, p, be determined by national supply and demand conditions. The f.o.b. price p grows at p^* per cent per year.

2. The price of capital goods to each firm in a given manufacturing industry is equal to $Z = Z_0 + ts$, where Z_0 is the uniform f.o.b. price and s is the distance from the producer. The variable t is transport cost per mile. The variable ts is assumed to have the same density function in each state for firms in a given manufacturing industry. Hence, the average c.i.f. price of capital goods, to firms in a given industry, will be the same for producers in each state.

Price Z_0 is determined by national supply and demand factors; the c.i.f. price grows by Z^* per cent for the average firm in each state, in a given industry.

3. Production functions of firms in a given manufacturing industry are assumed to be identical among states and subject to constant returns to scale. The uniformity of production functions is based on

[8] There is a formal equivalence between an explanation and a prediction. See Carl G. Hempel and Paul Oppenheim, "Studies in the Logic of Explanation," *Philosophy of Science*, Vol. 15, No. 2 (April, 1948).

[9] See Friedman, *op. cit.*

[10] E. Schneider, "Bemerkungen zu einer Theorie der Raumwirtschaft," *Econometrica*, 1935; R. G. D. Allen, *Mathematical Analysis for Economists* (London, 1949), pp. 80–82.

the view that interstate differences in soil fertility and climate are not very important in affecting manufacturing output, given the inputs of labor and capital.

Since we assumed that there are constant returns to scale for each firm, we must prevent a single firm from monopolizing the industry. Let us impose a limit to the size of each firm such that competition is allowed to prevail. Entry into this industry is free.

4. Finally, assume that pure competition prevails and firms attempt to maximize profits.

The assumptions enable us to expose the fundamental role of labor supply conditions in generating interstate differences in the rate of growth of manufacturing employment. The usefulness of the model and assumptions are examined in terms of the explanatory value of the model.

B. *Analysis.*

1. Every firm in a given manufacturing industry will select that ratio of capital to labor Q, such that the value of the marginal product of capital goods is equal to the price of capital goods. If f is the marginal-physical-product-of-capital function, p is the f.o.b. price of output, and Z is the c.i.f. price of capital goods, then

$$f(Q) = Z/p, \qquad f' < 0 . \tag{1}$$

It was assumed that each firm in the industry has the same f, which is a function solely of Q. In addition, p was assumed to be the same for all firms regardless of their regional location. Although the c.i.f. price of Z varies according to the purchaser's distance from the factory, the average value of Z will be the same from state to state in the given manufacturing industry. Consequently, *in long-run equilibrium*, Q will be the same for firms in the given industry regardless of their regional location.

Given the long-run equilibrium value of Q and the price of the product p, the marginal-physical-product-of-labor function F determines the equilibrium wage, equal to the value of the marginal product of labor:

$$pF(Q) = w , \qquad F' > 0 . \tag{2}$$

Insofar as p, F, and Q are, on the average, the same in each state in the given industry, the long-run equilibrium wage will be uniform among states.

Since the long-run equilibrium wage is determined, the labor supply function determines the quantity of labor that will be forthcoming and employed in each state.

$$(3) \qquad w = \frac{1}{v} g(L) ,$$

where v is the shift factor and L is the quantity of labor supplied. When v rises, the labor supply schedule shifts to the right, and when v falls, the labor supply schedule shifts to the left.

We solve for the state rate of growth of employment in the given industry, $L^* = \dot{L}/L$, in terms of three parameters. The parameters are the rate of growth of the product price, $p^* = \dot{p}/p$; the rate of growth of the average c.i.f. price of capital goods, $Z^* = \dot{Z}/Z$; and the shift of the state labor supply function, $v^* = \dot{v}/v$. Equations (1.1), (2.1) and (3.1) are derived by differentiating the logarithms of equations (1), (2) and (3) with respect to time:

$$(1.1) \qquad Z^* - p^* = \frac{f'(Q)\dot{Q}}{f(Q)} ,$$

$$(2.1) \qquad w^* - p^* = \frac{F'(Q)\dot{Q}}{F(Q)} ,$$

$$(3.1) \qquad w^* + v^* = \frac{g'(L)\dot{L}}{g(L)} .$$

Define the following four terms.

(a) $u = d \log Q/d \log f(Q) = \{f(Q)/f'(Q)\}(1/Q)$. This is the elasticity of the marginal-product-of-capital schedule with respect to Q and is negative.

(b) $e = d \log Q/d \log F(Q) = \{F(Q)/F'(Q)\}(1/Q)$. This is the elasticity of the marginal product of labor schedule with respect to Q and is positive.

(c) $s = \{g(L)/g'(L)\}(1/L)$. This is the elasticity of labor supply.

(d) $k^* = Z^* - p^*$. This is the growth of the "real" price of capital goods.

A substitution of (a)–(d) into (1.1)–(3.1) yields

$$(4) \qquad L_i^* = s_i\left(p^* + \frac{u}{e}k^*\right) + s_i v_i^* ,$$

where i refers to the ith state.

The rates of growth of the product price p^* and the real price of capital k^* are assumed to be the same in all states in a given manufacturing industry. Insofar as the production function is identical among firms in this industry, u/e is the same among states. Suppose that the elasticity of labor supply were uniform among states. Then,

$$(5) \qquad \frac{L_1^*}{L_2^*} = \frac{\{p^* + (u/e)k^*\} + v_1^*}{\{p^* + (u/e)k^*\} + v_2^*} .$$

Interstate differences in rates of growth of employment in a given manufacturing industry, *from one long-run equilibrium to another*, arise solely from interstate differences in the growth of the labor supply function.

2. It is well known that interstate differences exist in the average wage paid in given manufacturing industries. According to our model, this means that firms are not at their long-run equilibrium positions.[11] Firms in low-wage states will yield higher marginal products of capital than firms (in the same industries) located in high-wage states. The length of time required to bring the wage up to the equilibrium value depends upon the speed at which capital flows into low-wage areas to benefit from the higher marginal product of capital.

Suppose that the equilibrium wage rose by $y = p^* + (u/e)k^*$ per cent from (say) 1919 to 1953. If the wage in the low-wage state rose to the new equilibrium, then the per cent increase in the wage in the low-wage state would be $d(1 + y) + y$ per cent.[12] The variable d reflects the per cent difference between the equilibrium and the initial wage. In actuality, the wage may only rise by x per cent of that amount, due to the insufficiency of the capital inflow that occurred during the period. Consequently, the relative rate of growth of employment in the low-wage state to the high-wage state, in a given manufacturing industry, is[13]

$$(6) \qquad L_1^*/L_2^* = [y + xd(1 + y) + v_1^*]/[y + v_2^*] .$$

x is between zero and unity and reflects the extent to which the disequilibrium has been decreased.

It thus appears that the rate of growth of employment, in a given state in a given manufacturing industry, depends upon:

(i) the per cent increase in the equilibrium wage, y, which is the same for each firm in the given manufacturing industry,

(ii) the rate at which the labor supply schedule is shifting to the right in a given state, v^*,

(iii) the extent to which the wage was initially below the equilibrium

[11] According to our model $w = pF(Q)$. Since p, F, and Q are the same for firms in each state, long-run equilibrium implies wage equalization. Since $F' > 0$, given Q and p, a unique w results.

[12] Let w_0 be the initial equilibrium wage and w_0' be the new equilibrium wage, in both states. State 1 is the low-wage state where w_1 was initially below w_0, but state 2 initially paid the equilibrium wage, i.e., $w_2 = w_0$. Suppose that capital flows into both states and wage rates rise to w_0'.

In state 1, wage will rise by $(w_0' - w_1)/w_1$ and in state 2, wage will rise by $(w_0' - w_0)/w_0$. $(w_0' - w_1)/w_1 = \{(w_0' - w_0) + (w_0 - w_1)\}/w_1$. Let $(w_0' - w_0)/w_0 = y$ and $(w_0 - w_1)/w_1 = d$. Then, $(w_0' - w_1)/w_1 = d(1 + y) + y$.

[13] Assume that $s_1 = s_2$.

wage, i.e., d, and

(iv) the extent that the disequilibrium is eliminated, i.e., x.

Low-wage states, on the basis of (iii), would tend to have higher rates of growth of employment than high-wage states in the same industries, provided that migration did not occur from low- to high-wage states.

We shall now complete our model by discussing the determinants of v^*, the rate at which the labor supply curve shifts. There is evidence that out-migration occurs from low-wage to high-wage states. As a result, the low-wage states, on balance, need not have higher rates of growth of employment than high-wage states.

C. *The Determinants of the Shift of the Labor Supply Curve.*

The basic conclusion of the model is that interstate variations in rates of growth of manufacturing employment in a given industry stem from differences in the shift (v^*) of the labor supply schedule and from deviations (d) between the initial wage and the initial equilibrim wage. Interstate differences in the elasticity of labor supply (s) facing a given industry are ignored. Attention will now be focused solely upon interstate differences in v^*.

The shift, or growth, of the labor supply curve facing the manufacturing sector of a given state arises from several sources.

1. Intra-state migration from non-manufacturing employment to manufacturing employment will cause the labor supply schedule facing the manufacturing sector to shift.

It is well known that (1) the rate of natural increase of the labor force is greater in rural than in urban areas. Therefore, the rate of natural increase of the labor force is negatively associated with the degree of urbanization in a state. However, (2) the degree of urbanization is positively related to the per cent of the labor force employed in manufacturing. It follows that the rate of natural increase of the labor force is negatively related to the per cent of the labor force employed in manufacturing. The smaller the ratio of manufacturing to total state employment, the greater the potentialities for the growth of the manufacturing labor supply schedule through a migration of labor from non-manufacturing to manufacturing employment. In effect the ratio of non-manufacturing to total employment is used as a proxy for the labor force replacement rate,[14] i.e., the rate of natural increase

[14] The replacement ratio is defined as the ratio of the expected number of entrants into the age group 25–69 during a decade per 100 expected number of departures from this age group during the same decade through death or attaining retirement age. The replacement ratio was not used for the states because of the extensive calculation involved. In an extension of this study, these calculations will be made.

of the labor force.

Whether or not the labor force will shift from the non-manufacturing sector into the manufacturing sector depends upon the relative attractiveness of the manufacturing and non-manufacturing sectors. For example, a state may have 10 per cent of its employment in manufacturing, but the average income in manufacturing may be lower than the average (labor) income received in any other sector. The *incentive* for the intra-state shift from the non-manufacturing sector to the manufacturing sector would then be negative. Labor would tend to flow out of manufacturing into the higher average earning non-manufacturing sectors. The factors which determine the magnitude of the shift from the non-manufacturing to the manufacturing sector of a given state are: (a) the relative size of the non-manufacturing sectors as a proxy for the rate of natural increase of the labor force and (b) the relative attractiveness of the manufacturing sector compared to the non-manufacturing sectors.

The attractiveness of the manufacturing sector, for production worker employment, may be measured by the value of the average production worker's annual earnings in manufacturing relative to state per capita income. The average production worker's annual earnings represent the most likely amount that a production worker can earn in the manufacturing sector. This figure takes into account the average hourly earnings among the different manufacturing industries, the state composition of manufacturing industries, the opportunities for overtime work, and the steadiness of employment. State per capita income reflects the income received from all sources. The higher the ratio of the average production worker annual earnings to state per capita income, the greater the incentive for the labor force to shift into the manufacturing sector.

As a matter of historical evidence, the ratio of average yearly production worker earnings to state per capita income is negatively correlated with the ratio of manufacturing to total employment. In 1940, the correlation between these two variables was -0.65, among the forty-eight states. This is significantly different from zero at the one per cent level. Thus, the incentive to shift into the manufacturing sector appears to decrease as the ratio of manufacturing to total employment increases.

2. A second major source for v^* is interstate migration. The greater the rate of in-migration of people in the labor force age, as a percentage of the state labor force, the greater will be the rightward shift of the labor supply curve facing the manufacturing sector.

Migration occurs, to a large extent, from low-wage to high-wage states.[15] Hence, v^* is affected by the ratio of the wage in one state to the average wage prevailing in the other states.

In summary, the shift of the labor supply curve facing the manufacturing sector of a given state (v^*) is affected by four variables:

(a) the natural rate of increase, for which the ratio of non-manufacturing to total state employment is a proxy;

(b) the relative attractiveness of the manufacturing sector to the other sectors within the state;

(c) the rate of in-migration, which is affected by the ratio of the wage in this state to that prevailing in other states;

(d) non-economic factors, such as climate.

4. THE EXPLANATORY POWER OF THE THEORY

The theory implies the phenomena described in Part 1 above, and may therefore be considered an explanation of interstate differences in growth rates.

1. The theory implies that the rate of growth of manufacturing production worker employment, in a given manufacturing industry, will be higher in states with low ratios of manufacturing to total employment than in states with high ratios. This is compatible with the results of regression (A) and Table 2.

2. The theory implies that, when the manufacturing ratio is taken into account, the rate of in-migration will be positively associated with the rate of growth. This actually occurs, as is shown by regression (A) above. It is likely that the *partial* correlation of migration and growth reflects the fact that workers migrate from low-wage to high-wage areas. Consequently, the v^* in low-wage areas is less than one would expect on the basis of the rate of natural increase in the labor force; v^* in high-wage areas is greater than one would expect on the basis of the rate of natural increase.

3. The rate of growth of manufacturing production worker employment, in a given manufacturing industry, will not necessarily be higher in low-wage states than in high-wage states. The level of the state wage has two counterbalancing effects upon the rate of growth of employment.

(a) The marginal rate of return on capital is higher in low-wage states, in a given industry, than in high-wage states. Capital is there-

[15] See the work of Richard Muth and Harvey Perloff, *Variation in Levels and Rates of Growth of Per Capita Income*, Part IV of "Regional Growth in the United States," ch. 25 (Resources for the Future, 1959).

fore attracted to the state. As capital flows in, this generates a demand for labor. Thus, wages and growth rates should be correlated negatively.

(b) On the other hand, out-migration of labor occurs in low-wage states as labor flows from low- to high-wage states. As a result, out-migration tends to reduce v^* (the shift of the labor supply schedule facing manufacturing) in low-wage states. In high-wage states, in-migration of labor from low-wage states tends to increase v^*. This implication is compatible with the finding of regression (B), where the regression coefficient of x_3' was not significant.

4. The stability of relative growth rates, in terms of Table 1, is partially explained in terms of the invariance of the rankings of states by the per cent of employment in manufacturing. The composition of the ten states with the greatest ratios of manufacturing employment was practically unchanged from 1920 to 1940. Similarly, the composition of the ten states with the lowest ratios of manufacturing employment was practically unchanged during this period. The former group is dominated by states with no more than two successes (cf. Part 1), whereas there is no more than one state in the latter group which had fewer than four successes (cf. Part 1).

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BALANCED GROWTH OF FIRMS IN A COMPETITIVE SITUATION WITH EXTERNAL ECONOMIES*

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THE ASSUMPTION of independent production possibilities underlying the traditional general equilibrium theory implies that external economies and diseconomies do not prevail. More generally, however, in the presence of such economies and diseconomies, the level of activity in the individual firm affects the production possibilities of other firms through its influence on the general level of activity in ways other than through possible effects on the prices of goods and factors; that is to say, the production possibilities of a firm depend on the general development of industries. It is, therefore, very important to examine various effects of external economies and diseconomies on the development of firms.[2] In particular, it is interesting to compare the balanced rate of growth and the consumption level in a competitive situation in which such economies exist with the corresponding magnitudes in a situation in which these economies are absent. In the context of the model developed in this paper, we demonstrate (and this may be intuitively apparent) that the balanced rate of growth is higher in the former than in the latter case if the consumption level is the same, and that the highest allowable level of consumption is higher in the former than in the latter case if the rate of balanced growth is the same.

In the first section we describe a model in which external economies or diseconomies of the "unpaid-factor" type (in the sense of Professor Meade [5][3]) prevail. This is a mere generalization of the von Neumann model [2, 7] of balanced growth, which assumes the constancy of the technological coefficients, i.e., the absence of any external economies

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[2] The question of the relation between external economies and Pareto optimality is not discussed here, although it is surely of some interest. Under the assumption of no external economies E. Malinvaud [4] studied the correspondence between maximal and efficient balanced growth programs.

[3] Numbers in brackets refer to the list of references at the end.

or diseconomies. In the second section we are concerned with the existence of balanced growth solutions and show that, even if external economies and diseconomies exist, all firms may grow or decay proportionately. The less mathematically inclined reader may omit this section —— though the existence result is, of course, crucial and fundamental in the discussion of mathematical economics. In the third section we show how external economies react upon the rate of balanced growth of firms, and in the last section we compare the consumption level in the presence and absence of external economies.

1. THE ECONOMIC MODEL

We consider an economy in which there appear m firms (subscript i or r) and n goods (subscript j or s). It is assumed that there are a large number of competing firms, so that for each individual entrepreneur the market prices of the products and the factors are given. Let

$d_{ij}(t)$ be the quantity of good j produced by firm i in period t,

$e_{ij}(t)$ be the quantity of good j used up by firm i in period t,

$q_i(t)$ be the number of workers[4] employed by firm i in period t.

Define the following vectors:

$$d_i(t) = [d_{i1}(t), \cdots, d_{in}(t)],$$
$$e_i(t) = [e_{i1}(t), \cdots, e_{in}(t)],$$
$$d(t) = [d_{11}(t), \cdots, d_{1n}(t), \cdots, d_{m1}(t), \cdots, d_{mn}(t)],$$
$$e(t) = [e_{11}(t), \cdots, e_{1n}(t), \cdots, e_{m1}(t), \cdots, e_{mn}(t)],$$
$$q(t) = [q_1(t), \cdots, q_m(t)].$$

For the individual firm, the traditional production function may be written implicitly,

$$f_i[d_i(t),\ e_i(t),\ q_i(t)] = 0.$$

This assumes that inputs are transformed instantaneously into outputs, i.e., the period of production is zero. If we assume a unit production period, current outputs will depend on the inputs of the previous period, so we have

$$g_i[d_i(t+1),\ e_i(t),\ q_i(t)] = 0.$$

This is the case in the absence of external economies and diseconomies. In their presence, however, the current output of the ith firm will depend also on the activities of all firms. It is plausible to suppose that it will take time to diffuse the effects of the general development

[4] This paper assumes that labor is indispensable. This rules out the possibility of automatic factories.

of industries throughout the economy. Consequently, we postulate that the level of current output of the ith firm depends not on the current activities of all firms but on their activities in the previous period. It follows that the production function of the individual firm can be rewritten,

$$\phi_i[d_i(t+1),\ e_i(t),\ q_i(t);\ d(t),\ e(t-1),\ q(t-1)] = 0\ .$$

Professor Meade did not include this time lag, but assumed that the current activity of a firm would instantaneously affect the production possibilities of other firms.

On the other hand, we follow Professor Meade in distinguishing two different cases concerning the production function. For this, we write ϕ_i in explicit form as:

$$d_{i1}(t+1) = \psi_i[d_{i2}(t+1),\ \cdots,\ d_{in}(t+1),\ e_i(t),\ q_i(t);\ d(t),\ e(t-1),\ q(t-1)]\ .$$

The arguments which firm i can control in period t are $d_i(t+1)$, $e_i(t)$, and $q_i(t)$ only. Of course, it can control $d_i(t)$, $e_i(t-1)$, and $q_i(t-1)$ [which are included in the vectors $d(t)$, $e(t-1)$, and $q(t-1)$, respectively] in period $t-1$ but not in period t, so that $d(t)$, $e(t-1)$, and $q(t-1)$ play the role of parameters in the above function. We now distinguish between: (1) those cases in which $d_{i1}(t+1)$ is homogeneous of degree one in $d_{i2}(t+1),\ \cdots,\ d_{in}(t+1),\ e_i(t),\ q_i(t),\ d(t),\ e(t-1),\ q(t-1)$; and (2) those cases in which $d_{i1}(t+1)$ is homogeneous of degree one in $d_{i2}(t+1),\ \cdots,\ d_{in}(t+1),\ e_i(t)$, and $q_i(t)$, but not in all arguments including $d(t)$, $e(t-1)$, and $q(t-1)$. Meade calls the first case the "unpaid-factor" case and the second the case of the "creation of atmosphere."[5] In this paper, we are concerned with the former case only.

Let $p_j(t)$ be the price of good j in period t, $w(t)$ the money wage-rate in period t, and $\beta(t)$ the interest factor in the same period ($=1+$ the rate of interest in period t). All of them are given to each firm as determined in the market. Let $p_j^i(t+1)$ be the price of good j in period $t+1$ which is expected by firm i in period t. Suppose firm i has a given number of workers, $q_i(t)$, available for production in period t. Then the firm chooses the optimum factor-product combination, such that the capitalized value of profits

$$\frac{1}{\beta(t)}\left[\sum_{j=1}^{n} p_j^i(t+1)\,d_{ij}(t+1)\right] - \sum_{j=1}^{n} p_j(t)e_{ij}(t) - w(t)q_i(t)$$

is a maximum subject to the production function ϕ_i. This schedule

[5] See [3] p. 56 and p. 67.

gives $d_{ij}(t+1)$ and $e_{ij}(t)$ as functions of $p^e(t+1)$, $p(t)$, $\beta(t)$, $q_i(t)$, $d(t)$, $e(t-1)$, and $q(t-1)$. Now assume static expectations such that $p^e(t+1)=p(t)$. Since $d_{ij}(t+1)$ and $e_{ij}(t)$ are homogeneous functions of degree zero in $p^e(t+1)$ and $p(t)$, the functions under the assumption of static expectations can be written as:

$$d_{ij}(t+1)=f_{ij}[y(t),\ \beta(t),\ q_i(t),\ d(t),\ e(t-1),\ q(t-1)]\,,$$
$$e_{ij}(t)=g_{ij}[y(t),\ \beta(t),\ q_i(t),\ d(t),\ e(t-1),\ q(t-1)]\,,$$

where $y(t)=[y_1(t),\cdots,y_n(t)]$, and $y_s(t)=p_s(t)\Big/\sum_{j=1}^{n}p_j(t)$. Furthermore, from the assumption that ϕ_i is homogeneous in $d_i(t+1)$, $e_i(t)$, $q_i(t)$, $d(t)$, $e(t-1)$, and $q(t-1)$, we find that $d_{ij}(t+1)$ and $e_{ij}(t)$ determined above are homogeneous of degree one in $q_i(t)$, $d(t)$, $e(t-1)$, and $q(t-1)$; thus

$$\lambda d_{ij}(t+1)=f_{ij}[y(t),\ \beta(t),\ \lambda q_i(t),\ \lambda d(t),\ \lambda e(t-1),\ \lambda q(t-1)]$$

and

$$\lambda e_{ij}(t)=g_{ij}[y(t),\ \beta(t),\ \lambda q_i(t),\ \lambda d(t),\ \lambda e(t-1),\ \lambda q(t-1)]$$

hold for any value of scalar λ. Writing

$$d_{ij}(t+1)/q_i(t)\equiv b_{ij}(t+1)\,,$$
$$e_{ij}(t)/q_i(t)\equiv c_{ij}(t)\,,$$
$$q_i(t+1)/q_i(t)\equiv\alpha_i(t)\,,$$
$$q_r(t)\Big/\sum_{i=1}^{n}q_i(t)\equiv x_r(t)\,,$$

and putting $\lambda=1/q_i(t-1)$, we obtain:

$$\lambda d_{ij}(t+1)=d_{ij}(t+1)/q_i(t-1)\equiv\alpha_i(t-1)b_{ij}(t+1)\,,$$
$$\lambda e_{ij}(t)=e_{ij}(t)/q_i(t-1)\equiv\alpha_i(t-1)c_{ij}(t)\,,$$
$$\lambda d_{rj}(t)=d_{rj}(t)/q_i(t-1)\equiv b_{rj}(t)x_r(t-1)/x_i(t-1)\,,$$
$$\lambda e_{rj}(t-1)=e_{rj}(t-1)/q_i(t-1)\equiv c_{rj}(t-1)x_r(t-1)/x_i(t-1),$$
$$\lambda q_r(t-1)=q_r(t-1)/q_i(t-1)\equiv x_r(t-1)/x_i(t-1)\,.$$

Hence we find:

$$\alpha_i(t-1)b_{ij}(t+1)=f_{ij}[y(t),\ \beta(t),\ \alpha_i(t-1),\ b(t),\ c(t-1),\ x(t-1)]\,,$$
$$\alpha_i(t-1)c_{ij}(t)=g_{ij}[y(t),\ \beta(t),\ \alpha_i(t-1),\ b(t),\ c(t-1),\ x(t-1)]\,,$$

where $b(t)=[b_{11}(t),\cdots,b_{nn}(t)]$, $c(t-1)=[c_{11}(t-1),\cdots,c_{nn}(t-1)]$, and $x(t-1)=[x_1(t-1),\cdots,x_n(t-1)]$.

Let us now concentrate our attention on the state of balanced growth where $q_1(t),\cdots,q_n(t)$ grow or decay in the same proportion α, and

all the prices y, the growth factor $\alpha(=1+$ the rate of balanced growth), the interest factor β, and the technological coefficients b and c are constant over time. In that state,

$$\alpha_1(t) = \cdots = \alpha_m(t) = \alpha\,; \quad \beta(t) = \beta\,,$$

$$x(t) = x\,, \quad y(t) = y\,, \quad b(t) = b\,, \quad c(t) = c$$

hold for all t. It is known that under some suitable conditions there exists one and only one system of continuous functions

$$b_{ij} = b_{ij}(x,\ y,\ \alpha,\ \beta)\,,$$

$$c_{ij} = c_{ij}(x,\ y,\ \alpha,\ \beta)\,,$$

which satisfies the equations $\alpha b_{ij} - f_{ij}(y, \beta, \alpha, b, c, x) = 0$ and $\alpha c_{ij} - g_{ij}(y, \beta, \alpha, b, c, x) = 0$ $(i = 1, \cdots, m;\ j = 1, \cdots, n)$.[6] Note that the row vector[7] $x = (x_1, \cdots, x_m)$ with $x_r \geqq 0$, $\sum_{r=1}^{m} x_r = 1$ and the column vector $y = (y_1, \cdots, y_n)$ with $y_s \geqq 0$, $\sum_{s=1}^{n} y_s = 1$ denote the allocation of labor among firms and the *normalized* prices of goods, respectively.

Next we assume that capitalists only save[8] and workers only consume. Let $h_j(t)$ be workers' consumption of good j in period t. Then we have

$$w(t)\left[\sum_{i=1}^{m} q_i(t)\right] = p_1(t)h_1(t) + \cdots + p_n(t)h_n(t)\,,$$

or

$$w(t) = p_1(t)k_1(t) + \cdots + p_n(t)k_n(t)\,,$$

where $k_j(t) = h_j(t) \Big/ \sum_{i=1}^{m} q_i(t)$. According to the usual theory of consumer behavior, workers' demand for consumption goods allows substitution in response to price changes. In this paper, however, we assume that $k_1(t), \cdots, k_n(t)$ change proportionately, i.e., $k_j(t) = \gamma k_j$ $(j = 1, \cdots, n)$, where $k_1, \cdots, k_n$ are constant. The proportionality factor γ is referred to as the level of consumption. Writing $\omega(t) = w(t) \Big/ \sum_{s=1}^{n} p_s(t)$, we obtain

$$\omega(t) = \gamma k_1 y_1(t) + \cdots + \gamma k_n y_n(t)\,.$$

In the following two sections, γ is assumed constant and treated as a parameter in our system. The determination of the consumption level γ will be discussed in the last section of this paper.

[6] See [9] pp. 16–17.

[7] In our model, the vector x plays the same role as the activity vector x in the usual von Neumann models. Our vector x is referred to as the labor allocation vector.

[8] The case in which capitalists' average propensity to save is less than one is discussed in [6].

Let us now write:

$$a_{ij}(x, y, \alpha, \beta, \gamma) = c_{ij}(x, y, \alpha, \beta) + \gamma k_{ij},$$
$$A(x, y, \alpha, \beta, \gamma) = \| a_{ij}(x, y, \alpha, \beta, \gamma) \|,$$
$$B(x, y, \alpha, \beta) = \| b_{ij}(x, y, \alpha, \beta) \|.$$

In order for all firms to expand at a constant rate α, the following conditions must be satisfied:

$$(1) \qquad x[B(x, y, \alpha, \beta) - \alpha A(x, y, \alpha, \beta, \gamma)] \geqq 0,$$
$$(2) \qquad x[B(x, y, \alpha, \beta) - \alpha A(x, y, \alpha, \beta, \gamma)]y = 0;$$
$$(3) \qquad [B(x, y, \alpha, \beta) - \beta A(x, y, \alpha, \beta, \gamma)]y \leqq 0,$$
$$(4) \qquad x[B(x, y, \alpha, \beta) - \beta A(x, y, \alpha, \beta, \gamma)]y = 0,$$
$$(5) \qquad xB(x, y, \alpha, \beta)y > 0.$$

These conditions have the same interpretations as the analogous ones in the typical von Neumann models [2, 7], namely, (1) implies that it is impossible to consume more of a good than is being produced, (2) makes overproduced goods free, (3) makes firms profitless, (4) assures that unprofitable firms do not employ workers, and (5) requires that pricing and production schemes are such that something of value is produced.

To give a game-theoretic interpretation to these conditions let us imagine two fictitious economic persons whom we shall call the *producer* and the *pricer*. The producer controls the labor allocation vector x and actually is made up of many people;[9] in fact, all the entrepreneurs who are involved in production decisions in the economy make up, jointly, the producer. The pricer controls the price vector y and is also made up of many people, that is, all the people who participate in pricing decisions. As we shall see in the next section, it is possible to interpret the actions of the pricer and producer as being those of participants in a zero-sum two-person game. In fact, if the possible actions of these two players were to be spelled out in detail, one would obtain the extensive form of an "economic game" for which expressions (1)–(5) are the normalized form (cf. [3] or [8]).

2. THE EXISTENCE PROOF

As in [2], we need not expect equilibria to exist without making assumptions concerning A and B. The assumptions to be made here are the following:

[9] The device of treating many people having common interests as a single player is common in game theory (cf. [8]). Thus, bridge is considered a two-person game in which a player consists of two partners (e.g., North-South or East-West).

(i) *The functions* $a_{ij}(x, y, \alpha, \beta, \gamma)$ *and* $b_{ij}(x, y, \alpha, \beta)$ *are continuous for all* (x, y, α, β) *in the set*

$$S = \Big\{(x, y, \alpha, \beta) \mid x_i \geqq 0,\ \sum_{i=1}^{m} x_i = 1,$$

$$y_j \geqq 0,\ \sum_{j=1}^{n} y_j = 1,\ \sigma \geqq \alpha \geqq 0 \text{ and } \sigma \geqq \beta \geqq 0\Big\},$$

where σ *is a positive constant.*

(ii) *For all* (x, y, α, β) *in* S *the inequalities* $A(x, y, \alpha, \beta, \gamma) \geqq 0$, $B(x, y, \alpha, \beta) \geqq 0$, *and* $A(x, y, \alpha, \beta, \gamma) + B(x, y, \alpha, \beta) > 0$ *hold.*

(iii) *For all* (x, y, α, β) *in* S *the inequalities* $v[B(x, y, \alpha, \beta)] > 0$ *and* $v[B(x, y, \alpha, \beta) - \sigma A(x, y, \alpha, \beta, \gamma)] < 0$ *hold, where* B *and* $B - \sigma A$ *are considered as matrix games in which the maximizing player controls the rows and the minimizing player the columns, and* $v(B)$ *and* $v(B - \sigma A)$ *are the values of each of these games.*

Assumption (i) is a continuity assumption that prevents a_{ij} and b_{ij} from changing discontinuously from one point (x, y, α, β) to another, while Assumption (ii) is like the original von Neumann assumption in [7]. As for (iii), $v[B(x, y, \alpha, \beta) - \sigma A(x, y, \alpha, \beta, \gamma)] < 0$ merges into $v[-A(x, y, \alpha, \beta, \gamma)] < 0$ if σ is sufficiently large. $v[B(x, y, \alpha, \beta)] > 0$ and $v[-A(x, y, \alpha, \beta, \gamma)] < 0$ are like the assumptions made in [2].

In order to interpret the problem in game-theoretic terms we will need the following lemma:

LEMMA 1. *If* x, y, α *and* β *are solutions of* (1)–(5), *then* $\alpha = \beta$.

PROOF: From (5) we see that $xB(x, y, \alpha, \beta)y > 0$; hence from (2) and (4) we have $xB(x, y, \alpha, \beta)y = \alpha xA(x, y, \alpha, \beta, \gamma)y = \beta xA(x, y, \alpha, \beta, \gamma)y > 0$. From the last relation we have $xA(x, y, \alpha, \beta, \gamma)y > 0$ so that $\alpha = \beta = xB(x, y, \alpha, \beta)y / xA(x, y, \alpha, \beta, \gamma)y$.

The lemma tells us that we need look only for solutions in which the interest factor is equal to the growth factor. Under the assumption $\alpha = \beta$, equations (2) and (4) become identical and, in fact, are easy consequences of expressions (1) and (3) and the fact that x and y are non-negative vectors with unit sums. Writing $B(x, y, \alpha) = B(x, y, \alpha, \alpha)$ and $A(x, y, \alpha, \gamma) = A(x, y, \alpha, \alpha, \gamma)$ and making the abbreviation $M_\alpha(x, y, \alpha, \gamma) = B(x, y, \alpha) - \alpha A(x, y, \alpha, \gamma)$, we may write expressions (1), (3) and (5) as:

$$xM_\alpha(x, y, \alpha, \gamma) \geqq 0, \tag{1'}$$

$$M_\alpha(x, y, \alpha, \gamma)y \leqq 0, \tag{3'}$$

$$xB(x, y, \alpha)y > 0. \tag{5'}$$

These three expressions do not define a game, and it is not immediately clear how they should be interpreted. If, however, we let matrices A and B assume constant values, we can interpret this as an ordinary expanding economy model. In order to do this we choose x^*, y^*, and α^* such that $x_i^* \geqq 0$, $\sum_{i=1}^{m} x_i^* = 1$, $y_j^* \geqq 0$, $\sum_{j=1}^{n} y_j^* = 1$ and $\sigma \geqq \alpha^* \geqq 0$. Consider the following expressions:

$$(1'') \qquad xM_{\alpha}(x^*, y^*, \alpha^*, \gamma) \geqq 0\,,$$

$$(3'') \qquad M_{\alpha}(x^*, y^*, \alpha^*, \gamma)y \leqq 0\,,$$

$$(5'') \qquad xB(x^*, y^*, \alpha^*)y > 0\,,$$

where $M_{\alpha}(x^*, y^*, \alpha^*, \gamma) = B(x^*, y^*, \alpha^*) - \alpha A(x^*, y^*, \alpha^*, \gamma)$. Because of Assumptions (ii) and (iii) and the theorems in [2], [7], and [10], there is always a solution to these inequalities; moreover, the growth factor α is uniquely determined.[10] Let α^{**} be the unique solution; then we obtain

LEMMA 2. *α^{**} is positive and less than σ and depends continuously on the variables (x^*, y^*, α^*).*

PROOF: Let $(u_\theta^*, w_\theta^*, \lambda_\theta^*, \lambda_\theta^*)$ be any sequence of points in S that converge on the point $(x^*, y^*, \alpha^*, \alpha^*)$ and let λ_θ^{**} be the corresponding growth factors. Since the entries A and B are non-negative, $v(M_{\lambda_\theta})$ is a monotonic decreasing function of λ_θ. By Assumption (iii) λ_θ^{**}'s are positive and less than σ; thus λ_θ^{**}'s are contained in a bounded portion of the real line. Hence we can pick a convergent subsequence λ_τ^{**} with limit λ^{**}. The continuity assumption (i) implies that $\lim_{\tau\to\infty} A(u_\tau^*, w_\tau^*, \lambda_\tau^*, \gamma) = A(x^*, y^*, \alpha^*, \gamma)$ and $\lim_{\tau\to\infty} B(u_\tau^*, w_\tau^*, \lambda_\tau^*) = B(x^*, y^*, \alpha^*)$. Since $v[M_{\lambda_\tau^{**}}(u_\tau^*, w_\tau^*, \lambda_\tau^*, \gamma)] = 0$ for every τ implies $v[M_{\lambda^{**}}(x^*, y^*, \alpha^*, \gamma)] = 0$, we see that $\lambda^{**} = \alpha^{**}$; otherwise, the uniqueness of the growth factor [insured by Assumption (ii)] would be violated. If the sequence λ_θ^{**} itself does not converge, then we can pick two different convergent subsequences with different limits which would also violate uniqueness; hence $\lim_{\theta\to\infty} \lambda_\theta^{**} = \alpha^{**}$, and α^{**} depends continuously on the variables (x^*, y^*, α^*). Of course, $0 < \alpha^{**} < \sigma$.

Let us substitute α^{**} for α in (1''), (3'') and (5''), obtaining

[10] If the matrices A and B have constant entries and $M_{\alpha} = B - \alpha A$ satisfies the Assumptions (ii) and (iii) above, then by the results in [2] and [10] there is a solution to inequalities (1''), (3'') and (5''). Such a solution consists of an expansion factor α and a pair of vectors x and y that satisfy these inequalities. Finally, the result in [7] shows that the expansion factor found is unique.

(1‴) $$xM_{\sigma\sigma}(x^*, y^*, \alpha^*, \gamma) \geqq 0 ,$$

(3‴) $$M_{\sigma\sigma}(x^*, y^*, \alpha^*, \gamma)y \leqq 0 ,$$

(5‴) $$xB(x^*, y^*, \alpha^*)y > 0 .$$

Let $T(x^*, y^*, \alpha^*)$ be the following set:

$$T(x^*, y^*, \alpha^*) = \{(x, y) \mid \text{expressions (1‴) and (3‴) hold}\} .$$

We then obtain

LEMMA 3. *The mapping* $(x^*, y^*, \alpha^*) \to (T(x^*, y^*, \alpha^*), \alpha^{**})$ *is an upper semi-continuous mapping of the set*

$$U = \left\{(x, y, \alpha) \mid x_i \geqq 0, \sum_{i=1}^{m} x_i = 1, y_j \geqq 0, \sum_{j=1}^{n} y_j = 1 \text{ and } \sigma \geqq \alpha \geqq 0\right\}$$

into itself. Moreover, $T(x^*, y^*, \alpha^*)$ *is convex.*

PROOF: It follows from Lemma 2 that all the entries of the game $M_{\sigma\sigma}(x^*, y^*, \alpha^*, \gamma)$ depend continuously on the variables (x^*, y^*, α^*). Since the set of optimal strategies for a game varies in an upper semi-continuous way with the entries of the game, $(x^*, y^*, \alpha^*) \to (T(x^*, y^*, \alpha^*), \alpha^{**})$ is an upper semi-continuous mapping of S into itself. The convexity of T is obvious.

We can now establish

THEOREM 1. *A solution exists for inequalities* (1′), (3′) *and* (5′) *if Assumptions* (i), (ii) *and* (iii) *are satisfied.*

PROOF: By the Kakutani fixed point theorem [1] the mapping defined in Lemma 3 has a fixed point, say $(x^*, y^*) \in T(x^*, y^*, \alpha^*)$ and $\alpha^* = \alpha^{**}$. Then (x^*, y^*, α^*) satisfies (1′) and (3′), so we have

(6) $$x^*[B(x^*, y^*, \alpha^*) - \alpha^* A(x^*, y^*, \alpha^*, \gamma)]y^* = 0 .$$

By Assumption (ii) we also have

(7) $$x^*[B(x^*, y^*, \alpha^*) + A(x^*, y^*, \alpha^*, \gamma)]y^* > 0 .$$

Substituting from (6) into (7) we obtain

$$(1 + \alpha^*)x^*B(x^*, y^*, \alpha^*)y^* > 0 .$$

Therefore, we conclude that $x^*B(x^*, y^*, \alpha^*)y^* > 0$, which shows that (5′) holds, and we have established the existence of a state of balanced growth.

REMARK. So far we have assumed that σ is finite. If σ is infinite, Assumptions (i), (ii) and (iii) are replaced by:

(i′) *The functions* $a_{ij}(x, y, \alpha, \beta, \gamma)$ *and* $b_{ij}(x, y, \alpha, \beta)$ *are continuous for all* (x, y, α, β) *in the set*

$$S' = \Big\{(x,\ y,\ \alpha,\ \beta) \mid x_i \geqq 0,\ \sum_{i=1}^{n} x_i = 1,$$
$$y_j \geqq 0,\ \sum_{j=1}^{n} y_j = 1,\ \alpha \geqq 0 \text{ and } \beta \geqq 0\Big\}.$$

These functions approach certain limits as both α and β become infinite, and those limits are continuous for all pairs of non-negative vectors x and y with unit sums.

(ii') *For all (x, y, α, β) in S' the inequalities $A(x, y, \alpha, \beta, \gamma) \geqq 0$, $B(x, y, \alpha, \beta) \geqq 0$, and $A(x, y, \alpha, \beta, \gamma) + B(x, y, \alpha, \beta) > 0$ hold, and* $\lim_{\alpha,\beta\to\infty} [A(x, y, \alpha, \beta, \gamma) + B(x, y, \alpha, \beta)] > 0$.

(iii') *For all (x, y, α, β) in S' the inequalities $v[-A(x, y, \alpha, \beta, \gamma)] < 0$ and $v[B(x, y, \alpha, \beta)] > 0$ hold, and* $\lim_{\alpha,\beta\to\infty} v[-A(x, y, \alpha, \beta, \gamma)] \neq 0$ *and* $\lim_{\alpha,\beta\to\infty} v[B(x, y, \alpha, \beta)] \neq 0$.

By the same method as above we can prove

THEOREM 2. *If Assumptions* (i'), (ii') *and* (iii') *hold, the system* (1)-(5) *has a solution $(x^*, y^*, \alpha^*, \beta^*)$ such that $(x^*, y^*, \alpha^*, \beta^*) \in S'$ and $\alpha^* = \beta^* =$ finite.*

3. EXTERNAL ECONOMIES AND THE RATE OF BALANCED GROWTH

In this section we are concerned with the effect of external economies upon the rate of balanced growth of firms. In the absence of any external economies, we have the simplest (independent) production functions:

$$(8)\quad F_i[b_{i1}(t+1), \cdots, b_{in}(t+1), c_{i1}(t), \cdots, c_{in}(t)] = 0 \quad (i = 1, \cdots, m),$$

assuming that there are constant returns to scale in each firm and that the production period is unity. In equilibrium the capitalized value of profits per worker,

$$\frac{1}{\beta(t)}\Big[\sum_{j=1}^{n} p_j^*(t+1)b_{ij}(t+1)\Big] - \sum_{j=1}^{n} p_j(t)c_{ij}(t) - w(t),$$

is at a maximum subject to the production function (8), so that, under the assumption of static expectations, $b_{ij}(t+1)$ and $c_{ij}(t)$ depend on $y(t)$, and $\beta(t)$. If we are restricted to the case of balanced growth in which prices and the rate of interest are constant over time, we obtain

$$b_{ij} = b_{ij}(y,\ \beta),$$
$$c_{ij} = c_{ij}(y,\ \beta).$$

Let us write: $a_{ij}(y, \beta, \gamma) = c_{ij}(y, \beta) + \gamma k_{ij}$, $A(y, \beta, \gamma) = \| a_{ij}(y, \beta, \gamma) \|$, and $B(y, \beta) = \| b_{ij}(y, \beta) \|$. The expressions corresponding to (1)-(5) in Section 1 now become

$$x[B(y,\ \beta) - \alpha A(y,\ \beta,\ \gamma)] \geqq 0\ , \tag{9}$$

$$x[B(y,\ \beta) - \alpha A(y,\ \beta,\ \gamma)]y = 0\ , \tag{10}$$

$$[B(y,\ \beta) - \beta A(y,\ \beta,\ \gamma)]y \leqq 0\ , \tag{11}$$

$$x[B(y,\ \beta) - \beta A(y,\ \beta,\ \gamma)]y = 0\ , \tag{12}$$

$$xB(y,\ \beta)y > 0\ . \tag{13}$$

Under conditions similar to (i)–(iii) all of the work in Section 2 holds, and the existence theorem insures that the system (9)–(13) has at least one economic solution $(\bar{x},\ \bar{y},\ \bar{\alpha},\ \bar{\beta})$.

It is plausible enough to assume that any factor-product combination, feasible in the absence of external economies, is still feasible under external economies, but no longer efficient. Under this assumption we can obtain

THEOREM 3. *The presence of external economies gives rise to an increase in the rate of balanced growth.*

PROOF: Let x^0, y^0, α^0, and β^0 be the solution of (1)–(5). The profits with the chosen output- and input-coefficients, $b_{ij}(x^0,\ y^0,\ \alpha^0,\ \beta^0)$ and $c_{ij}(x^0,\ y^0,\ \alpha^0,\ \beta^0)$, when normalized prices are y^0 and the interest factor is β^0, should be greater than the profits which could have been made had any other feasible, but inefficient, combination been chosen, so that

$$\begin{aligned}&\frac{1}{\beta^0}\sum_{j=1}^{n} b_{ij}(\bar{y},\ \bar{\beta})y_j^0 - \sum_{j=1}^{n} c_{ij}(\bar{y},\ \bar{\beta})y_j^0 - \omega \\ &\quad < \frac{1}{\beta^0}\sum_{j=1}^{n} b_{ij}(x^0,\ y^0,\ \alpha^0,\ \beta^0)y_j^0 - \sum_{j=1}^{n} c_{ij}(x^0,\ y^0,\ \alpha^0,\ \beta^0)y_j^0 - \omega\ , \\ &\qquad\qquad (i = 1, \cdots, n)\ .\end{aligned} \tag{14}$$

This follows, since the combination

$$[b_{i1}(\bar{y},\ \bar{\beta}),\ \cdots,\ b_{in}(\bar{y},\ \bar{\beta}),\ \cdots,\ c_{i1}(\bar{y},\ \bar{\beta}),\ \cdots,\ c_{in}(\bar{y},\ \bar{\beta})]$$

is still feasible, but not efficient, when there are external economies. Together with $\sum_{j=1}^{n} \gamma k_j y_j^0 = \omega$, (14) leads to:

$$\begin{aligned}&B(\bar{y},\ \bar{\beta})y^0 - \beta^0 A(\bar{y},\ \bar{\beta},\ \gamma)y^0 \\ &\qquad < B(x^0,\ y^0,\ \alpha^0,\ \beta^0)y^0 - \beta^0 A(x^0,\ y^0,\ \alpha^0,\ \beta^0)y^0\ .\end{aligned} \tag{15}$$

From (3) it follows that the right-hand side of (15) does not exceed zero. Hence

$$B(\bar{y},\ \bar{\beta})y^0 < \beta^0 A(\bar{y},\ \bar{\beta},\ \gamma)y^0\ .$$

Multiply this by $\bar{x}$, obtaining $\bar{x}B(\bar{y}, \bar{\beta})y^0 < \beta^0\bar{x}\, A(\bar{y},\bar{\beta}, \gamma)y^0$; and multiply (9) by y^0, obtaining $\bar{x}B(\bar{y}, \bar{\beta})y^0 \geqq \bar{\alpha}\bar{x}A(\bar{y}, \bar{\beta}, \gamma)y^0$; these two expressions imply that $\beta^0 > \bar{\alpha}$. Since $\beta^0 = \alpha^0$, we get $\alpha^0 > \bar{\alpha}$, completing the proof.

4. DETERMINATION OF THE CONSUMPTION LEVEL

So far we have treated the consumption level γ as a parameter and determined the x^0, y^0, α^0 and β^0 corresponding to the γ. In this section we are concerned with two situations with the same rate of growth of population, in one of which there are external economies, while in the other they are absent; the consumption levels in these situations are compared with each other.

We begin by establishing the following lemma:

LEMMA 4. *If the consumption level tends to infinity, the rate of balanced growth tends to* -1.

PROOF: It follows from (2) that

$$xB(x, y, \alpha, \beta)y = \alpha xC(x, y, \alpha, \beta)y + \gamma\alpha \sum_{j=1}^{n} k_j y_j .$$

Since (x, y, α, β) belongs to the set S, and since $B(x, y, \alpha, \beta)$ is continuous for all (x, y, α, β) in S, $xB(x, y, \alpha, \beta)y$ does not exceed a properly chosen constant. On the other hand, $xC(x, y, \alpha, \beta)y$ is non-negative. Hence α must tend to zero as γ tends to infinity.

Let us now assume that the system (1)–(5) determines a unique rate α corresponding to a given γ. Then it is easily seen that the α is a

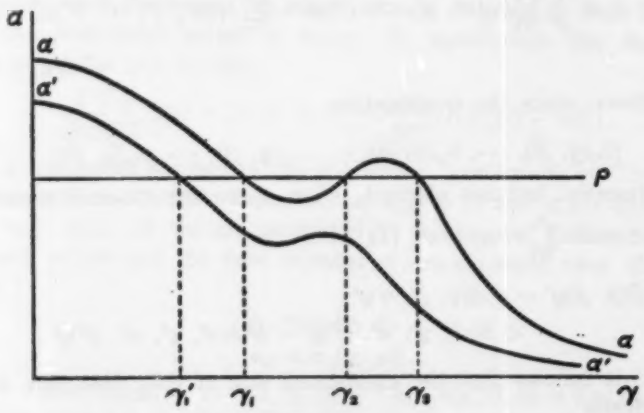

FIGURE 1

continuous function of γ. As Lemma 4 holds, the function $\alpha(\gamma)$ has a shape like curve $\alpha\alpha$ in Figure 1. Curve $\alpha'\alpha'$ stands for the function $\alpha(\gamma)$ in the absence of external economies. By virtue of Theorem 3, it must be located below $\alpha\alpha$.

Suppose the supply of labor in period t is $\rho^t N$, where the rate of growth of population $\rho - 1$ is assumed constant. Since the demand for labor in period t is $\alpha^t \sum_{i=1}^{m} q_i(0)$, the demand-supply balance of labor is described as:

$$\alpha^t \sum_{i=1}^{m} q_i(0) = \rho^t N \qquad \text{for all } t\,,$$

which is equivalent to the following two equations:

$$\sum_{i=1}^{m} q_i(0) = N$$

$$\alpha = \rho\,.$$

If $\max \alpha(\gamma) \geqq \rho$, there exists at least one γ such that $\alpha(\gamma) = \rho$, cf. Figure 1. But if $\max \alpha(\gamma) < \rho$, then even if $\sum_{i=1}^{m} q_i(0) = N$, we will have

$$\alpha^t \sum_{i=1}^{m} q_i(0) < \rho^t N \qquad \text{for } t \geqq 1\,.$$

Hence unemployment of labor will be inevitable, and we would expect that the rate of real wages (and hence the consumption level γ) will be pushed down to the subsistence level. (In any case, the consumption level cannot be determined within our model for this situation.) The balanced growth associated with the greatest γ among those which make α and ρ equal may be called the golden balanced growth, and that γ may be referred to as the highest allowable level of consumption or the golden level of consumption. The theorem which follows is immediately obvious from Figure 1.

THEOREM 4. *The highest allowable level of consumption in the presence of external economies is higher than that in their absence.*

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THE KEYNESIAN THEORY OF INFLATION: THE TWO FACES OF JANUS?*

BY SIDNEY WEINTRAUB

CONSIDERING THE high unanimity with which writers of the Keynesian stamp approach problems of total employment and aggregate output, it is somewhat odd and baffling that on the great practical problem of the decade now closing, specifically, the inflation issue, their views have split them into separate camps of "demand-pull" and "cost-push," or some uneasy amalgam of the two, as with varying intensity they have examined the always partial and inconclusive empirical evidence. On the inflationary subject, I think that differences have tended to submerge likenesses, so that we have an unaccustomed array of opponents in the intellectual arena.

A "school" in dispute is obviously an interesting spectacle and must expect to draw attention to its plight. At the same time, the split intellectual personality that I think I detect on this one great issue is undoubtedly capable of generating new contradictions over time on other problems of greater or lesser importance.

In this paper I am not mainly concerned with the question of which of the two major inflation doctrines best fits the facts, or whether both should not be scrapped in favor of renewed emphasis on monetary factors and a Fisher-type equation of exchange. I have expressed myself elsewhere on this general subject and have little more to add to it at this time.[1] What concerns me is a more fascinating theoretical, or methodological and philosophical matter, to wit, why economists who agree that the cure for unemployment is to use remedies stemming from the theory of the consumption function, the investment function, and the rate of interest, should find themselves at loggerheads and arrive at an impasse when it comes to an analysis of inflation and the price level. It is the origin of this Janus-head on inflation, the two voices from one source, that I should like to unravel in the following pages. Perhaps greater clarity on the theoretical plane will help sharpen the important issues with which the present generation of economists must learn to cope.

* Manuscript received January 7, 1960.

[1] See my *A General Theory of the Price Level, Output, Income Distribution and Economic Growth* (Philadelphia: Chilton Company, 1959).

At the outset I should like to state firmly that I believe this "muddle in the middle" is more of the making of Keynesians than of Keynes himself; I think I "know" what Keynes would have emphasized, but I refrain from any attempt at documentation, for it is equally possible for those who hold alternative views to find appropriate supporting passages in Keynes' *General Theory* or his later discussion in *How to Pay for the War* to sanction their doctrinal interpretation. If nothing else, this demonstration of different brands of Keynesianism, which have often been thought of as fully substitutable wares, discloses that there may be new rifts brewing under the quiet facade which may establish divergences in directions currently unsuspected.

One further prefatory remark is in order. I have refrained from identifying proponents of the conflicting views and from identifying them with analytic error for several reasons. In the first place, some parts of the very imposing theoretical edifice have now become common property at the elementary and teaching level, so that, if I am right, the error is widespread. Second, much of this apparatus was constructed without this special problem in view, so that some interpretation is necessary; it is a hard matter to disentangle the price level analyses embodied in even the most rigorous versions of the theory. Third, it has happened that with what I regard as the natural apparatus of "excess-demand" and "demand-pull"—using these terms synonymously—some of the technicians may, and do, adopt a "cost-push" posture in approaching public policy, and *vice versa*. None of us are entirely free of such contradictions within our theoretical structures, and so more than the usual pains would have been required to make sure that the personifications were not unfair. In any event, it is the split personality in ideas among the school, not the "error" of individuals, that I want to explore.[2]

1. THE 45° LINE AND THE KEYNESIAN EQUILIBRIUM CROSS

The dominant brand of Keynesian analysis, it can be said without fear of contradiction, runs in terms of the 45-degree line bisecting the coordinate field containing the familiar and assorted paraphernalia of consumption function and investment magnitude. I shall examine this diagram in a moment and, as it can be found in practically every modern-day textbook, the examination can be brief. It is this ap-

[2] Originally, as I began to document this paper, I found that the footnote discussion was far outrunning the textual material, and to little or no purpose. On reconsideration and in asking constantly whether the minor thumbnail analyses were fair to the authors I decided to present merely the major ideas.

paratus, with its associated real income equilibrium determination, which has been compared in importance with the demand and supply curves of price theory, and the equilibrium "cross" of the one identified as equivalent in principle to the equilibrium intersection of the other.

Used somewhat less frequently, and emanating from Professor Hicks' masterly exposition many years ago, are the pair of *IS* and *LM* curves which are supposed to give more generality to the argument by showing the simultaneous macroeconomic adjustment of aggregate output and the rate of interest.[3] Both approaches—and both are really the same for my present purposes, for each rests on the same real output basis—represent the main wing of neo-Keynesian analysis and are equally regarded as Keynesian economics, pure and simple or profound and unadulterated. They have come to occupy a complete teaching tradition, at least in this country, and perhaps through the world, in view of the world-wide dissemination of American textbooks. The ideas may well constitute a fairly universal version of Keynesian economics.

In Figure 1 there is drawn the diagram of the 45° cross which

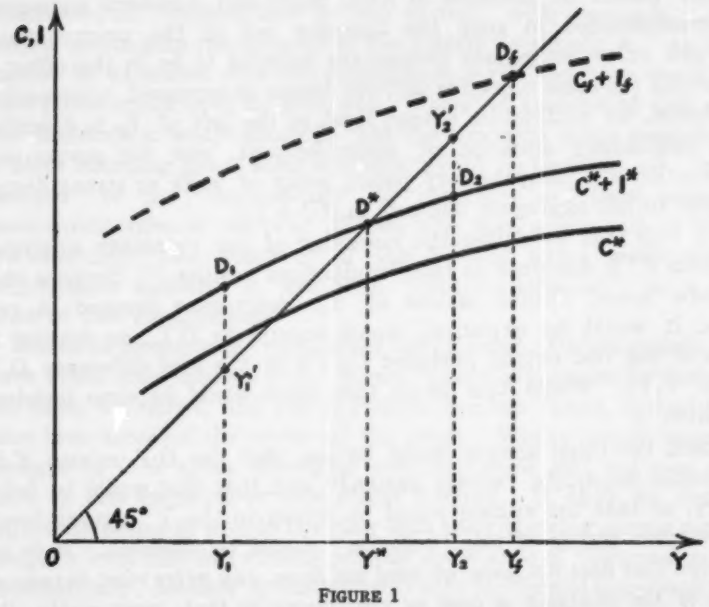

FIGURE 1

[3] Professor Hansen has probably taught us most about this approach which still remains above the elementary textbook level. See his *Monetary Theory and Fiscal Policy* (New York: McGraw-Hill Book Co., Inc., 1949), Chapter 5.

typifies this tradition. The 45° bisector represents the potential equilibrium relationships in a diagram in which real income or real output, measured horizontally and denoted by Y, is associated with real outlay in the form of consumption expenditures (C) and investment outlay (I), also in real terms, and measured vertically. Note that all quantities are in real terms; that is, C, I, and Y are money sums measured in constant dollars. Changing price level phenomena are thus completely eliminated from the picture; real magnitudes alone provide the focus of attention.

The diagram, as drawn, contains the now commonplace figure of the consumption function, lettered C^*, and has superimposed on it a constant sum of real investment (as in the more typical expositions) to comprise the real outlay function ($C^* + I^*$). The equilibrium intersection is at a volume of real output equal to Y^*. Here $Y^* = C^* + I^*$, so that macroequilibrium prevails.

Suppose that full employment output is located at position Y_f. Clearly, an aggregate demand ($C^* + I^*$) is inadequate and real output at Y^* is too low, carrying unemployment with it. Usually, the familiar gamut of measures of fiscal policy and monetary maneuvers are recommended to snap the economy out of the unemployment doldrums unless exogenous factors are believed to be in the offing to accomplish the same objective without deeper government intervention. In general, an equilibrium output level to the left of Y_f is described as a deflationary situation of unemployment, ripe for government intervention and compensatory public policy of weak or strong degree in order to lift aggregate demand to ($C_f + I_f$).

It is well that we follow the reasoning of the "stability analysis" by which Y^* is described as the "equilibrium position." Suppose that somehow actual output settled at Y_1. Aggregate demand in real terms, it would be explained, would amount to D_1Y_1, an amount in excess of the real output produced (Y_1Y_1') by the real difference D_1Y_1 *minus* Y_1Y_1'. Would this mean that there would be some incipient "inflation"?

I think the usual answer would be *yes*, that the D_1Y_1 minus Y_1Y_1' gap would constitute "excess demand" and that this would be inflationary, so that the system would be driven to the Y^* output level. The peculiar nature of this "inflation" should be observed. *It is an inflation that does not have, or need not have, any price rise, inasmuch as all of the analysis is cast in real terms so that, presumably, the price level at Y_1 is the same as at Y^*, at Y_2, at Y_f, etc. Price levels are excluded from the analysis*, as being in some way incidental,

so that only through some back-door approach can they be brought in. The only valid interpretation of the disequilibrium position at Y_1 is that if output temporarily settled here in some "round" of production, there would be "inflation" in the sense of inventory decumulation at the constant price level, equal in amount to the D_1Y_1' discrepancy. The run-down of inventory holdings would constitute an expansionary phenomenon, and a production advance toward the equilibrium "cross" would be in order.[4] Likewise, at an output level of Y_2 there would be an "unwarranted" or "unintended" inventory accumulation of $Y_2'D_2$, so that pressures would be apparent to drive real output back to Y^*. Deflationary forces would thus start the output cutback and unemployment train.

Other than the minor "inflation" analyzed as part of the stability analysis, the interesting question is how a truly inflationary process would be analyzed by Keynesians using this approach. I think that the majority of Keynesians would argue that in inflation what must be happening is a parametric rise in the $C + I$ curve to a position well above the "full employment equilibrium" function $C_f + I_f$, drawn as a dashed line in Figure 1, so that true "excess-demand" and "demand-pull" materializes in the system, with demands for real output outrunning the capacity of the system to satisfy it. As output cannot be expanded once full employment is reached, they would argue that prices would have to give way. From an analysis of this nature the Keynesians would conclude that the necessary policy steps to avoid a more alarming price level outbreak would be to choke off "excess-demand" or the "demand-pull" by monetary and fiscal means, or some combination of the two. (Those who take a less serious view of the durability and buoyancy of the demand forces would probably countenance learning to live with "creeping-inflation," as a minor price of full employment.)

Where do money wage rates enter the picture? Unfortunately, these have often gone undiscussed. Money wages are assumed constant in the basic argument, and the aggregate demand forces, in real terms, have been accorded the center of the stage. Where money wages are brought in *it is often merely to aver that these will not rise unless there is full employment and excess demand.* If they do rise, the argument seems to lean to the view that their upswing merely supports the price level that has already jumped up because of "excess-demand." Schematically, because it fits so badly into the formal apparatus which

[4] It is surprising to learn that often there is no discussion of the "stability conditions." The stronger versions do run in the terms indicated, and properly so, of inventory accumulation and decumulation at nonequilibrium positions.

runs in real terms, when this group is pressed on the subject, some rather gratuitous remarks on the effect of money wage changes on the distribution of income and hence on the level of aggregate demand are often tendered. But such effects are generally accorded rather meager attention.

In sum, as inflation is a creature of excess-demand, there is no serious price level distortion until full employment is reached. Similarly, the money wage level may be disregarded or accepted as a datum. Typically, exogenous wage movements through union pressure and business acquiescence are ignored until full employment. And then, it is the demand level that is stressed and creates the illusion of the "cost-push," as this school sees it.

The more sophisticated version of Keynesianism of this same stamp, with claims to more generality and insight, seeks to show the simultaneous determination of interest rate and real income level through the interdependence of the self-same functions. Thus in Figure 2 there is the well-known *IS* curve, which is built out of the real investment response to interest rate changes, the consumption function, and the $C + I$ identity, and is associated with various levels of real income or real output measured along the horizontal axis. In deriving *IS* it is the interest rate which is assumed to be the parametric constant, variations in which affect investment and thus the *IS* path. The *LM* curve is erected out of the remaining equations of Keynesianism, namely, the liquidity function and the money supply, regarded or-

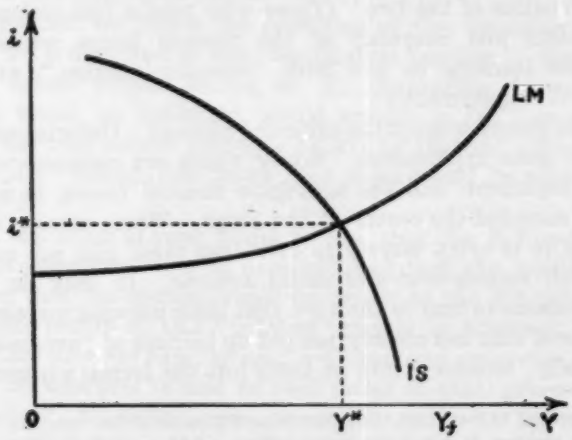

FIGURE 2

dinarily as a datum. In deriving the *LM* path it is the income level—*real* income[5]—which is temporarily viewed as the parameter. The diagrammatic apparatus thus permits the disclosure of the mutual income-interest rate values that are simultaneously equilibrating. In Figure 2 the equilibrium values are i^* and Y^*. Once again, if full employment real income is at Y_f, achievement of this zenith will require parametric movements of the two functions, through consumption, investment, or interest rate developments.

Until "full employment," in neither of these versions of Keynesianism—which are regarded as equivalent for my purposes—is there any substantial concern with inflation, *for all its concepts are cast in real terms*. The income levels being talked about have no explosive price levels, nor even changing price levels, for all this is assumed away. Surely, there is no concern with money wage changes—the "cost-push"—for the very application of the technique precludes this discussion: it is competent to handle variations in real phenomena, not in monetary phenomena. There is no way in which it can fit in, say, added monopoly power in the sense of discussions of advancing levels of administered prices where these are deemed important. For both changing monopoly power and changing money wages at full employment or less have their impact on price levels rather than on real output. But these are the very problems which have been exorcised by this technique which runs in real terms.

As one reflects on it, it becomes ever more astonishing that diagrams such as those examined have been regarded as useful in illuminating inflation when their very essence consists of conceptions running in real terms, thereby eliminating the very problem of changing price levels, whether due to new issues of money, or velocity changes, or money wage changes, or changes in productivity, or monopoly power—or anything else. This wing of Keynesianism, whose hallmark is either the analysis surrounding Figures 1 or 2, or other similar versions running in real terms, has justifiably been charged as devoid of a basic explanation of inflation.[6] I think this criticism is merited, for

[5] I have criticized the inclusion of the interest rate and the *real*—not money—income as the strategic variables in the liquidity function, and have suggested that the price level must also be included as a major parameter. See my *Price Level*, *op. cit.*, p. 72. I have been surprised at certain protests to the effect that the price level has *always* been included in the money-demand equation. I submit that an examination of the literature does *not* bear this out though in common discussion economists will invariably acknowledge its importance.

[6] Richard T. Selden, "Cost-Push versus Demand-Pull Inflation, 1955-57," *Journal of Political Economy*, (February, 1957), p. 5n.

in the last two decades, since 1940, the economy can be viewed as operating under conditions of strong aggregate demand, so that levels of real demand have not been in issue, and yet the *numéraire* phenomenon of changing money wage levels which fit uncomfortably in a "real" scheme has been barging upwards. It is little wonder that analyses springing from the Equation of Exchange have become more acceptable today than in the immediate heyday of Keynesianism.

Lest it be thought that I am dealing only with straw men picked only from the softer geometrical treatments presented to the neophyte students of economics, an examination of some mathematical models of the same stripe and in the recognized sources discloses exactly the same arguments; there is no inconsistency between mathematical and geometrical versions on this score, although I do not rule out the possibility that more elaborate Keynesian models may deal more adequately with price level phenomena. In those systems that take enough pains to include price level analyses rather than working with real phenomena right from the start, the most common hypothesis is that the money wage level is constant, with the usual reservation that this holds only until "full employment" obtains. What happens to the price level along the course of the associated C, I, or savings functions until full employment is reached is literally never discussed in these accounts, according to my experience in examining the standard works. The typical assumption seems to be that prices hold constant, or variations are so small that they may be neglected, until full employment is reached. Unless this proviso is mentally inserted, there is no simple way of interpreting the geometrical figures like Figures 1 and 2 that usually accompany the equational statements. One less frequently quoted equational system makes the money wage a function of the level of employment. Apart from the problems (undiscussed in the original) with respect to "money illusion," this simple function makes the point better than I would expect, for it leads inevitably to the argument that the high levels of money wages occur only in the vicinity of full employment: it would have no way of explaining why, in recent years, with little or no change in employment levels the money wage keeps mounting.

Thus I think I have presented fairly the dominant ideas of "demand-pull" Keynesian analysis on the inflation issue by way of the formal, and now not so formidable, diagrammatic and mathematical apparatus.

2. AGGREGATE MONEY DEMAND AND AGGREGATE MONEY SUPPLY FUNCTIONS

Let us now turn to the other face of Janus, to look instead at an alternate diagrammatic approach which, although not as commonly employed as the 45° line, still has a literature of its own. As I have used this apparatus in several writings over the past few years, the exposition of the main points can be made briefly.[7] The "cost-push" argument comes out of it quickly, effortlessly, and directly.

In Figure 3 *uncorrected* money levels of actual and expected expenditure are measured vertically, and amounts of labor are measured horizontally. All labor is assumed to be homogeneous, initially at least; this is a simplification not worse (more acceptable in my view) than the use of homogeneous units of real output (Y) in the dominant Keynesian approach. The curve Z, as drawn, represents an aggregate supply function, and it is premised on the short-period (Keynesian) assumption of a given stock of capital equipment, and it also assumes all along its course that the money wage level is held constant: this last hypothesis is tantamount to Keynes' analysis in terms of a given wage unit or money wage structure. As the curve is drawn,

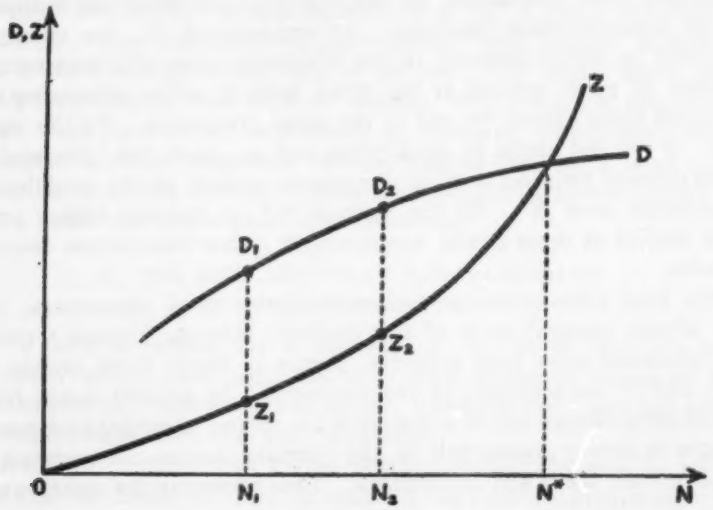

FIGURE 3

[7] See my *An Approach to the Theory of Income Distribution* (Philadelphia: Chilton Company, 1958), Chapter 2. For references to the earlier literature on the Z-function, see p. 24.

rising at an increasing rate to the right, it reflects some implicit diminishing returns phenomena and thus some necessary rising prices (of moderate proportions) as employment advances. Each point on the Z-function is to be interpreted as signifying the volume of outlays expected by firms in the economy in the form of consumer expenditures and investment outlays, which will lead the enterprises to give the associated level of employment.[8] Thus at an expected outlay level of Z_1 the employment that firms will create is N_1, etc. Manifestly, by writing the concomitant production function we can tether each employment level to a particular real output level, as N_1 to Y_1, etc.

On the same diagram we can draw the aggregate demand function (D). This shows the *actual* outlays, as distinct from the necessary or expected outlays along the Z-function, associated with each N-level and the implicit money wage payments to labor. Further, at each demand point, as at D_1, precisely the same price level is built into this point as is embodied in the corresponding Z_1 point at the N_1 amount of labor. At employment N_2, say, as prices rise because of the phenomena of diminishing returns while money wages hold stable, the higher price level implicit at point Z_2 is also subsumed in the associated D_2 expenditure sum.

Consider what this means for the stability analysis or the examination of nonequilibrium positions. At employment N_1, for example, there will be excess demand, in the Walrasian sense of microeconomic markets, of goods wanted at the given level of prices outrunning the quantities being offered for sale at the same price scale. To the right of N_1, as we get closer to equilibrium and as prices rise, the level of excess demand dwindles until it disappears entirely at the equilibrium employment level N^*. To the right of N^*, at the still higher price levels implicit at these points, excess supply rather than excess demand obtrudes.

Price level assumptions, and changing price level phenomena, are thus always integral parts of this analysis: whereas Figures 1 and 2 have exorcised price level concepts, Figure 3 keeps them always in mind and in the forefront of the discussion. In general, apart from technological change and/or a change in the volume of capital equipment, changes in money wages will be the primary reason for parametric shifts in both the Z and D functions. Thus whenever the money wage

[8] This would suggest that the Z-function should be decomposed into these components. But exactly the same observation may be made with respect to the 45° line approach. For an early recognition of this, see Dudley Dillard, *The Economics of John Maynard Keynes* (New York: Prentice-Hall, Inc., 1948), p. 35n.

changes, and whatever the reason for the change, from this diagrammatic apparatus it becomes instantly clear that: (a) the price level *must* change, and (b) the money income or gross national product (Z) in money terms must change, while (c) employment and real output *may* change only if the new intersection of D and Z functions occurs at a point other than above N^*. If the employment level is affected by the money wage change—and Keynes himself was largely a skeptic on this point—then it should be possible to derive a theory of money wages by aid of this apparatus.[9]

I think this very brief exposition will suffice for my purpose. From the analysis surrounding Figure 3 it can immediately be argued that even if money wages were held constant there would still be price level movements of greater or lesser degree, due to the laws of returns (and changing degrees of monopoly power). More important for the great controversies of the last decade or so, the argument leads inexorably to the conclusion that the prime mover in price level fluctuations, or the factor of overriding importance in discussions of inflation, is the wage level and incipient changes in it. This is the heart of the "cost-push" thesis: it devolves naturally and inevitably from this diagrammatic approach in contrast to its submersion into utter obscurity in the 45° line story. Excess demand, or a rise in the D-function in Figure 3, when unaccompanied by a money wage change, is seen to have a much less significant impact on the price level, which then merely responds to the laws of return. Conversely, a rise in the money wage level either as a result of changing employment levels or because of sheer exogenous factors, such as the need of labor union leaders to meet the desires of their membership, is shown always to have a clear and direct price level incidence.

3. THE BASIS AND POLICY CONSEQUENCES OF THE DIVERGENCE IN VIEWS

It is well to reflect for a moment on how it is that the two Keynesian wings have come up with such divergent views on the inflation problem, with one stressing the "demand-pull" and the other the money wage level and the "cost-push."

For myself, I think that at the bottom of the difference in views there lies the choice of working units by the separate groupings, with the one thinking largely in terms of real output or money amounts

[9] I originally adopted this apparatus for just this purpose, a theory of *money* wage determination. See my article, "A Macroeconomic Approach to the Theory of Wages," *American Economic Review*, (December, 1956).

corrected for price level changes, while the other has regarded the money wage as the appropriate *numéraire* and the employment unit as the bed-rock of analysis. On this interpretation it is the difference in the choice of deflators that seems to have been responsible for the mischief. While we may agree with Professor Hansen in principle when he wrote that "fundamentally the matter [of the choice of units, of real output or money wage] is of no great consequence,"[10] it seems to me that this very matter has led to a great practical cleavage with vast policy implications. For when one thinks of real output as basic, it is an easy step to envisage inflation as ensuing when real output can no longer be increased. On the other hand, when wage units are recognized as the *numéraire*, the value of the money wage is recognized as crucial in deciding the level of prices.

One approach thus leans inevitably to policy recommendations designed to hold wage requests in check. The other school, the dominant Keynesian branch, argues in terms of containing the level of demand to rein in prices, and thereby money wages. But here is the crucial anomaly which must sometimes rend this school, and the conflicting groups, even further: how far must real demand be cut back to control inflation? Must actual unemployment be created? If so, how much? If unemployment is not undertaken to discipline labor, can destabilizing wage movements be prevented?

If unemployment is the answer to the inflation problem, then Keynesianism as a social philosophy is dead, literally interred by Keynesians and, curiously, all in the name of the mentor. For to restrain inflation its ideas must be used—perverted, I think—to *create*, not abolish, unemployment. My own belief is that Lord Keynes' spirit must somewhere be stirring with more than a little discomfiture at the ultimate irony of this strange twist to his ideas.

I think that this great antithesis has been recognized, though not in this form, by several who approach Keynesianism in terms of aggregate demand and "demand-pull," and are chary about wage controls because they do not fit readily into their philosophical and analytical systems of thought. I refer to the great emphasis of this group on the role of economic growth as the answer to inflation, and their willingness to tolerate what is now fashionable to call "creeping inflation." But growth, if it brings with it exorbitant push-ups in

[10] Alvin H. Hansen, *A Guide to Keynes* (New York: McGraw-Hill Book Co., Inc., 1953), p. 44. In this point, as in so many others, Professor Hanson has written with a refreshing breadth and candor on a matter which most other writers since Keynes' day have not even bothered to discuss.

wages, will disappoint their hopes for merely "creeping inflation," and so the issue will ultimately have to be faced. After all, we have had tremendous growth since, say 1940, and yet, more than a doubling of the price level. Growth, thus, is hardly an answer to the inflation problem.

The only alternative to conscious and deliberate creation of unemployment in an economy in which the forces of aggregate demand are running strong, and one which could still preserve the analytical and philosophical system fashioned by Lord Keynes, would be some deliberate design and control of the money wage level, just as we have learned to use fiscal and monetary policy to control aggregate demand. Only this could permit the social philosophy and moderate policy intervention attitude of Keynes to survive in a world of reasonably full employment. Otherwise, Keynesianism would have to be intentionally reversed, with the intellectual framework originally created to eradicate the curse of unemployment deliberately and perilously distorted, to be used as an analytical engine to foster unemployment, so that the other evil, permanent inflation as a way of life, is repressed in favor of more tolerable standards of price level stabilization. How to contain the money wage level while maintaining full employment, and without breeding unacceptable interferences in labor markets, comes close to being the most important policy issue of our day.[11]

University of Pennsylvania, U. S. A.

[11] Among Keynesians who have given what I think to be the appropriate emphasis to the money wage as the dominant element in inflation, I would cite Professor A. P. Lerner who has described "seller's inflation" in a number of papers (e. g., *Hearings*, Joint Economic Committee, May 1958, p. 128), T. Balogh, "Productivity and Inflation," *Oxford Economic Papers*, (June, 1958), and Nicholas Kaldor, "Inflation and Economic Growth," *Economica*, (November, 1959), especially pp. 292–296.

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